Duration: **50 minutes**
Aids Allowed: **NONE** (in particular, no calculator)

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): ________________________________

First (Given) Name(s): ________________________________

*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 3 questions on 10 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, write your student number where indicated at the bottom of every odd-numbered page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do — part marks *will* be given for showing that you know the general structure of an answer, even if your solution is incomplete.

If you are unable to answer a question (or part), you will get 20% of the marks for the question (or part) if you write "I don't know" and nothing else — you will *not* get those marks if your answer is completely blank, or if it contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

Marking Guide

# 1:	______/10
# 2:	______/10
# 3:	______/13
Bonus Marks:	______/ 6
TOTAL:	______/33

Good Luck!

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 1. [10 MARKS]

Part (a) [3 MARKS]

State the Principle of Well Ordering.

Part (b) [3 MARKS]

State the Principle of Complete Induction in symbolic form, for an arbitrary predicate $P(n)$.

Part (c) [4 MARKS]

Explain how the principle of complete induction allows us to conclude that $P(1)$ is true unconditionally.

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 2. [10 MARKS]

Fill in each proof structure below so that it is correct, *i.e.*, so that the conclusion follows from the proof.

Part (a) [4 MARKS]

Predicate: $P(n)$.

Base Case(s):

Ind. Hyp.: Suppose $n \geqslant 1$ and

Ind. Step:

Conclusion: $\forall n \in \mathbb{N}, n$ is a power of 3 $\implies P(n)$.

Part (b) [3 MARKS]

Predicate: $P(n)$.

Ind. Hyp.:

Ind. Step: Prove $P(n)$.

Conclusion: $\forall n \geqslant 2, P(n)$.

Part (c) [3 MARKS]

Predicate: $P(n)$.

Base Case(s):

Ind. Hyp.: Suppose $n \in \mathbb{Z}$ and $P(n)$.

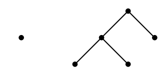
Ind. Step: Prove $P(n+1)$ and $P(n-1)$.

Conclusion:

Use this page for rough work — clearly indicate any section(s) to be marked.

Question 3. [13 MARKS]

Recall that the *height* of a binary tree is defined as the maximum number of nodes on any path from the root of the tree to one of its leaves. For example, the first tree on the right (with a single node) has height 1 and the second tree has height 3 — the empty tree (with no node) has height 0.



Prove that for all $h \in \mathbb{N}$, every binary tree of height h contains at most $2^h - 1$ nodes. Keep in mind that your proof will be marked on its structure at least as much as on its content, so write it up carefully.

Use this page for rough work — clearly indicate any section(s) to be marked.

Bonus. [6 MARKS]

WARNING! This question is difficult and credit will only be given for making *significant* progress toward a correct answer — in particular, you will NOT get 20% for writing "I don't know". Please attempt this only after you have completed the rest of the test.

Write a detailed proof structure that could be used to prove $\forall n \in \mathbb{Q}, P(n)$ ($\mathbb{Q}$ is the set of rational numbers), and explain briefly why your proof structure is correct.

On this page, please write nothing except your name.

Last (Family) Name(s): ______________________________

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