

Duration: **50 minutes**  
 Aids Allowed: **NONE** (in particular, no calculator)

Student Number:

Last (Family) Name(s):

First (Given) Name(s):

*Do **not** turn this page until you have received the signal to start.*  
 (In the meantime, please fill out the identification section above,  
 and read the instructions below *carefully*.)

This term test consists of 4 questions on 12 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, write your student number where indicated at the bottom of every page (except page 1), and write your name on the back of the last page.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of a page and *indicate clearly the part of your work that should be marked.*

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, assignments, or in the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

If you are unable to answer a question (or part of a question), you will get 20% of the marks for the question (or part of the question) if you state clearly that you do not know how to answer. Note that you will *not* get those marks if your answer contains contradictory statements (such as “I don’t know” followed or preceded by parts of a solution that have not been crossed off).

MARKING GUIDE

# 1: \_\_\_\_\_/12

# 2: \_\_\_\_\_/ 7

# 3: \_\_\_\_\_/ 8

# 4: \_\_\_\_\_/ 8

BONUS

MARKS: \_\_\_\_\_/ 5

TOTAL: \_\_\_\_\_/35

*Good Luck!*

*Use this page for rough work.*

**Question 1.** [12 MARKS]**Part (a)** [6 MARKS]

The “characteristic function” of a language  $L$  over alphabet  $\Sigma$  is defined as follows:

$$\text{for all } w \in \Sigma^*, \quad \chi_L(w) = \begin{cases} 1 & \text{if } w \in L, \\ 0 & \text{if } w \notin L. \end{cases}$$

Prove that for all languages  $L$ ,  $L$  is decidable iff  $\chi_L$  is computable.

**Part (b)** [6 MARKS]

Let  $\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \dots$  be the list of all TMs in lexicographic order. Define a real number  $r = 0.d_1d_2d_3\dots$  as follows (where  $d_1, d_2, d_3, \dots$  are the digits in the decimal expansion of  $r$ ):

$$\text{for } i = 1, 2, 3, \dots \quad d_i = \begin{cases} 1 & \text{if } L(M_i) = \emptyset, \\ 0 & \text{if } L(M_i) \neq \emptyset. \end{cases}$$

Prove or disprove that  $r$  is a constructible real number, where the definition of “constructible” was given in Assignment 1. (HINT: Let  $I$  be a TM that, on input  $\langle M \rangle$ , computes the index  $i$  such that  $\langle M \rangle = \langle M_i \rangle$ ; in your answer, you may make use of  $I$ .)

*Use this page for rough work.*

**Question 2.** [7 MARKS]

Prove or disprove each statement below, where  $L_1 = \{0^n 1^n : n \geq 0\}$ . (HINT: Think about the definition of  $A \leq_m B$  and about the consequences of proving  $A \leq_m B$  on the relative decidability of  $A$  and  $B$ .)

**Part (a)** [4 MARKS]

Prove or disprove:  $L_1 \leq_m A_{TM}$ .

**Part (b)** [3 MARKS]

Prove or disprove:  $A_{TM} \leq_m L_1$ .

*Use this page for rough work.*

**Question 3.** [8 MARKS]

Prove that  $\text{LARGE} = \{ \langle M, k \rangle : |L(M)| \geq k, \text{ i.e., } M \text{ accepts at least } k \text{ different inputs} \}$  is recognizable. (Write up your solution carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know how to answer this type of question, even if your solution is not complete.)

*Use this page for rough work.*

**Question 4.** [8 MARKS]

Prove that  $\text{SMALL} = \{ \langle M, k \rangle : |L(M)| \leq k, \text{ i.e., } M \text{ accepts no more than } k \text{ different inputs} \}$  is unrecognizable. (HINT: Use  $\leq_m$ . Write up your solution carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks *will* be given for showing that you know how to answer this type of question, even if your solution is not complete.)

*Use this page for rough work.*

**Bonus.** [5 MARKS]

**WARNING!** This question is difficult and credit will only be given for making significant progress toward a correct answer—in particular, you will NOT get 20% for writing “I don’t know”. Please attempt this only after you have completed the rest of the test.

Prove that  $INF = \{ \langle M \rangle : L(M) \text{ contains infinitely many strings} \}$  is unrecognizable.

Do not write anything on this page, except your name.

Last (Family) Name(s): \_\_\_\_\_

First (Given) Name(s): \_\_\_\_\_

Total Marks = 35