Duration: **50 minutes**
Aids Allowed: **NONE** (in particular, no calculator)

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name: ______________________________

First (Given) Name(s): ______________________________

Tutorial Section:
(cirle one)

GB–404	BA–1210	BA–2135	PA–70
Anna Popivanova	Yinghua Jia	Tovi Grossman	Anna Bretscher

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This term test consists of 4 questions on 5 pages (including this one), printed on one side of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete.*

Answer each question directly on the test paper, in the space provided, and use the reverse side of the pages for rough work. If you need more space for one of your solutions, use the reverse side of the page and *indicate clearly the part of your work that should be marked.*

If you are unable to answer a question (or part of a question), you will get 20% of the marks for the question (or part of the question) if you state clearly that you do not know how to answer. Note that you will *not* get those marks if your answer contains contradictory statements (such as "I do not know how to answer" followed or preceded by parts of a solution that have not been crossed off).

General Hint: We were careful to leave ample space on the test paper to answer each question.

Marking Guide

1: ______/10

2: ______/15

3: ______/14

4: ______/ 6

Bonus Marks: ______/ 1

TOTAL: ______/45

Good Luck!

Question 1. [10 MARKS]

Part (a) [2 MARKS]

If $b_{k-1}b_{k-2}\dots b_1b_0$ is a binary representation for $n \in \mathbb{N}$, give a binary representation for $2n$.

Part (b) [8 MARKS]

State whether each proof structure below is correct or not. If incorrect, explain what error(s) it contains.

- **Base Case:** Prove $P(1)$.
  Ind. Hyp.: Assume $P(k)$ is true for all $k \geq 1$.
  Ind. Step: Prove $P(k+1)$.
  Conclusion: $P(n)$ is true for all $n \geq 1$.

- **Base Case:** Prove $P(1)$ and $P(2)$.
  Ind. Hyp.: For some $k > 2$, assume $P(k-1)$.
  Ind. Step: Prove $P(k)$.
  Conclusion: $P(n)$ is true for all $n \geq 1$.

- **Base Case:** Prove $P(1)$.
  Ind. Hyp.: For some $k > 1$, assume $P(k)$.
  Ind. Step: Prove $P(k+1)$.
  Conclusion: $P(n)$ is true for all $n \geq 1$.

- **Base Case:** Prove $P(2)$.
  Ind. Hyp.: For some $k \geq 1$, assume $P(k)$.
  Ind. Step: Prove $P(k-1)$ and $P(k+2)$.
  Conclusion: $P(n)$ is true for all $n \geq 1$.

Question 2. [15 MARKS]

Consider the sequence of numbers $r_1, r_2, \ldots$ defined as follows:

$$r_1 = \sqrt{2},$$
$$r_n = \sqrt{2 + r_{n-1}} \qquad \text{for } n > 1.$$

Prove that $r_n < 2$ for all $n \geq 1$. (Note that approximately half of the marks for this question will be awarded solely for a correct proof structure.)

Question 3. [14 MARKS]

Part (a) [7 MARKS]

Prove or disprove the following logical equivalence:

$$p \leftrightarrow (q \rightarrow \neg p) \iff p \wedge \neg q$$

Part (b) [7 MARKS]

Prove or disprove the following statement:

$$\forall x p(x) \vee \neg \exists x p(x) \text{ is valid}$$

Question 4. [6 MARKS]

For each formula below, state whether or not it is in Prenex Normal Form, then write a logically equivalent formula in which every bound variable has a unique name that is also different from all of the free variables. (Note: do *not* put the formula in PNF, only rename bound variables. Also, you do not have to justify the equivalence of your formula to the original formula.)

Part (a) [2 MARKS]

$\forall x \exists z (P(x, y) \rightarrow Q(y, z))$

Part (b) [2 MARKS]

$\forall x (P(x, y) \wedge \exists x (Q(z, x) \rightarrow \forall y S(y)))$

Part (c) [2 MARKS]

$\forall x P(x, y) \rightarrow Q(z, x)$

Bonus. [1 MARK]

Write your student number at the bottom of every page of this test except page 1, where indicated. Also, if you have not done so already, complete the identification section at the top of page 1.

Total Marks = 45