CSC 411: Lecture 03: Linear Classification

Class based on Raquel Urtasun & Rich Zemel's lectures

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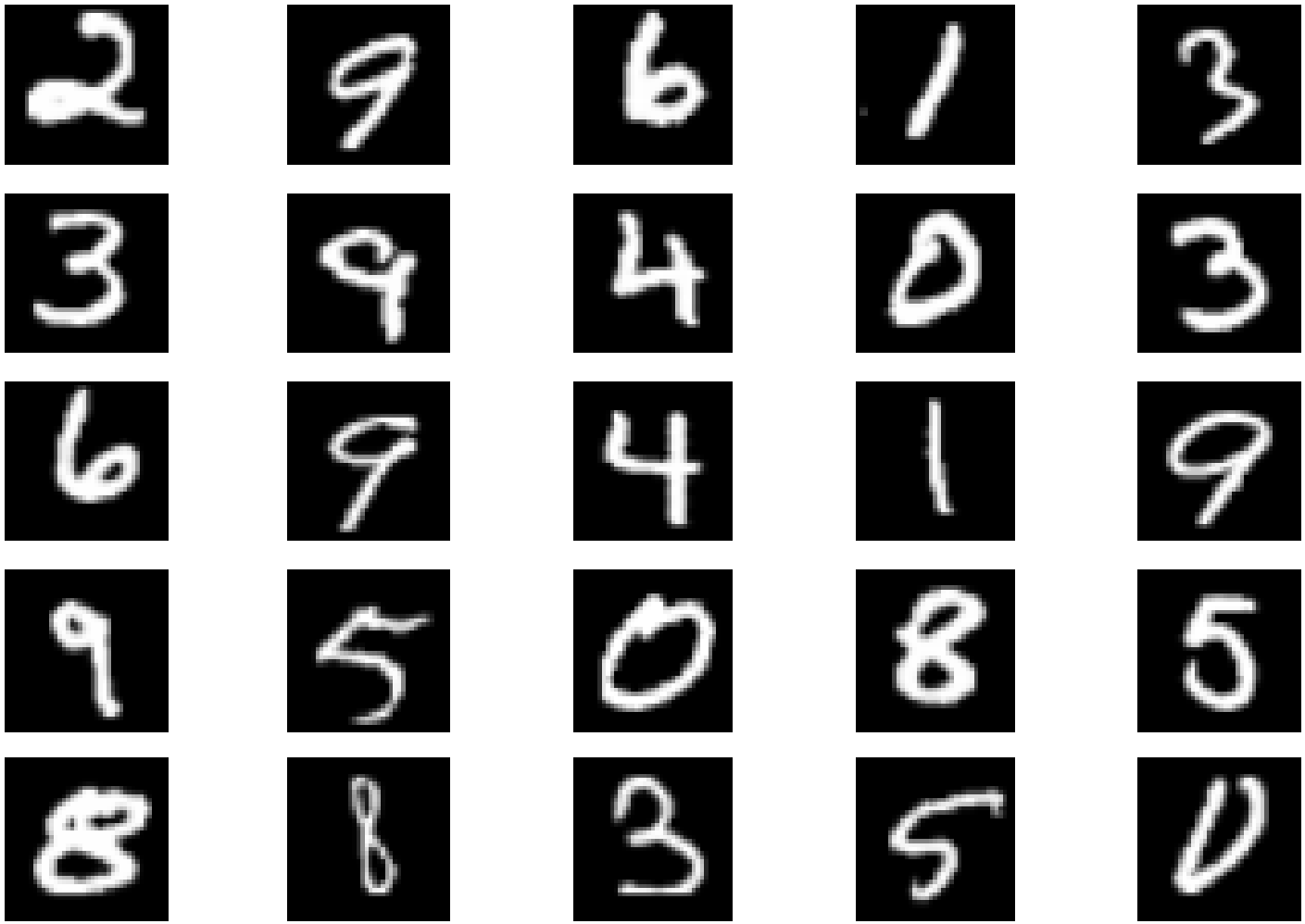
Examples of Problems



What digit is this?

How can I predict this? What are my input features?

Regression

- What do all these problems have in common?
- Categorical outputs, called labels
  (eg, yes/no, dog/cat/person/other)
- Assigning each input vector to one of a finite number of labels is called classification
- **Binary classification**: two possible labels (eg, yes/no, 0/1, cat/dog)
- **Multi-class classification**: multiple possible labels
- We will first look at binary problems, and discuss multi-class problems later in class

Today

- Linear Classification (binary)
- Key Concepts:
  - Classification as regression
  - Decision boundary
  - Loss functions
  - Metrics to evaluate classification

Classification vs Regression

- We are interested in mapping the input $\mathbf{x} \in \mathcal{X}$ to a *label* $t \in \mathcal{Y}$
- In regression typically $\mathcal{Y} = \Re$
- Now $\mathcal{Y}$ is categorical

Classification as Regression

- Can we do this task using what we have learned in previous lectures?
- Simple hack: Ignore that the output is categorical!
- Suppose we have a binary problem, $t \in \{-1, 1\}$
- Assuming the standard model used for regression

$$y = f(\mathbf{x}, \mathbf{w}) = \mathbf{w}^T \mathbf{x}$$

- How can we obtain $\mathbf{w}$?
- Use least squares, $\mathbf{w} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{t}$. How is $\mathbf{X}$ computed? and $\mathbf{t}$?
- Which loss are we minimizing? Does it make sense?

$$\ell_{square}(\mathbf{w}, t) = \frac{1}{N} \sum_{n=1}^{N} (t^{(n)} - \mathbf{w}^T \mathbf{x}^{(n)})^2$$

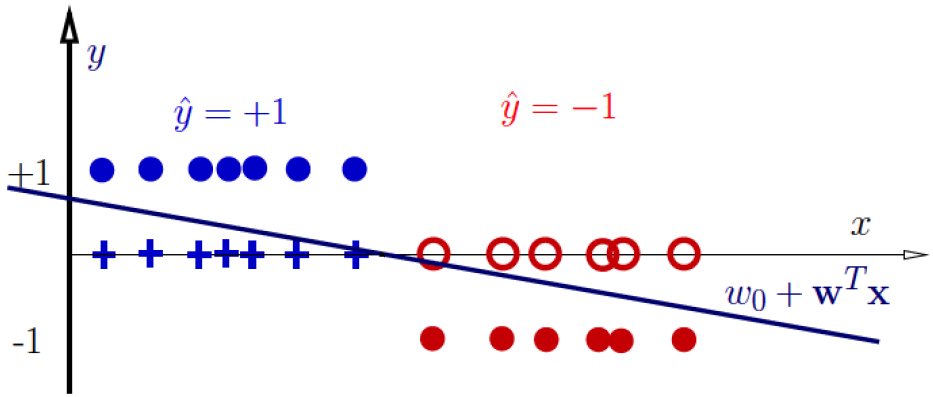
- How do I compute a label for a new example? Let's see an example

Classification as Regression

- One dimensional example (input x is 1-dim)

- The colors indicate labels (a blue plus denotes that $t^{(i)}$ is from class -1, red circle that $t^{(i)}$ is class 1)

Decision Rules



- Our classifier has the form

$$f(\mathbf{x}, \mathbf{w}) = w_o + \mathbf{w}^T\mathbf{x}$$

- A reasonable decision rule is

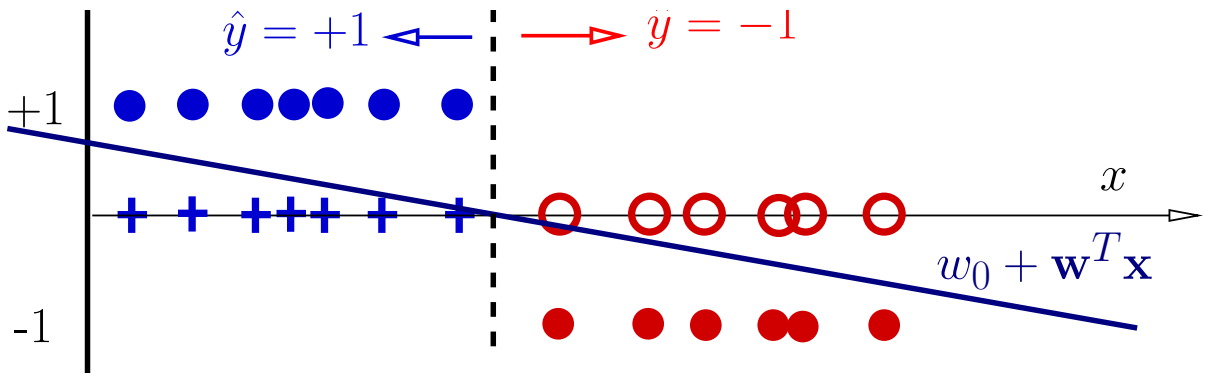
$$y = \begin{cases} 1 & \text{if } f(\mathbf{x}, \mathbf{w}) \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

- How can I mathematically write this rule?

$$y = \text{sign}(w_0 + \mathbf{w}^T\mathbf{x})$$

- How does this function look like?

Decision Rules



- How can I mathematically write this rule?

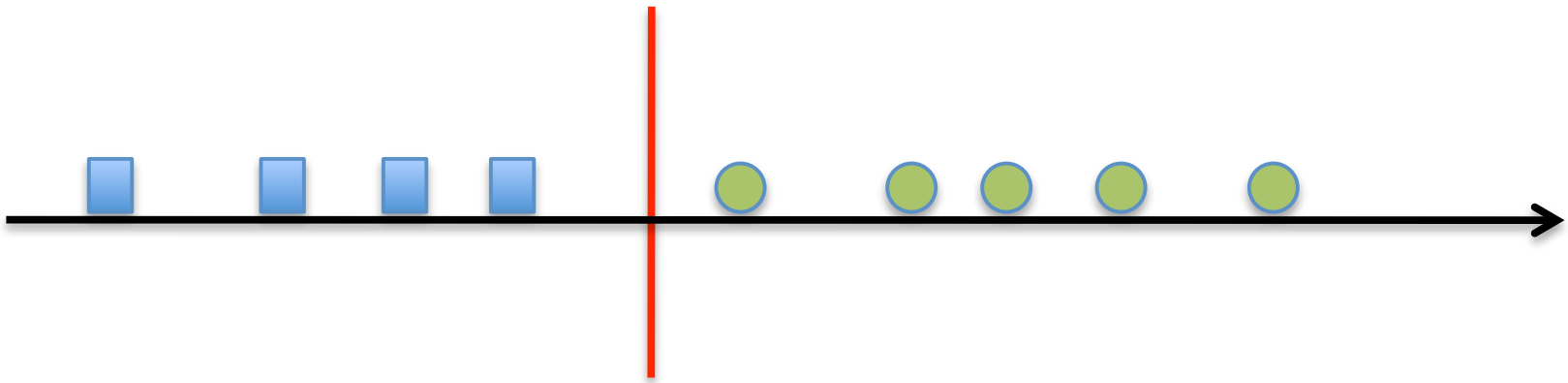
$$y = \text{sign}(w_0 + \mathbf{w}^T\mathbf{x})$$

- This specifies a linear classifier: it has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T\mathbf{x} = 0$$

which separates the space into two "half-spaces"

Example in 1D

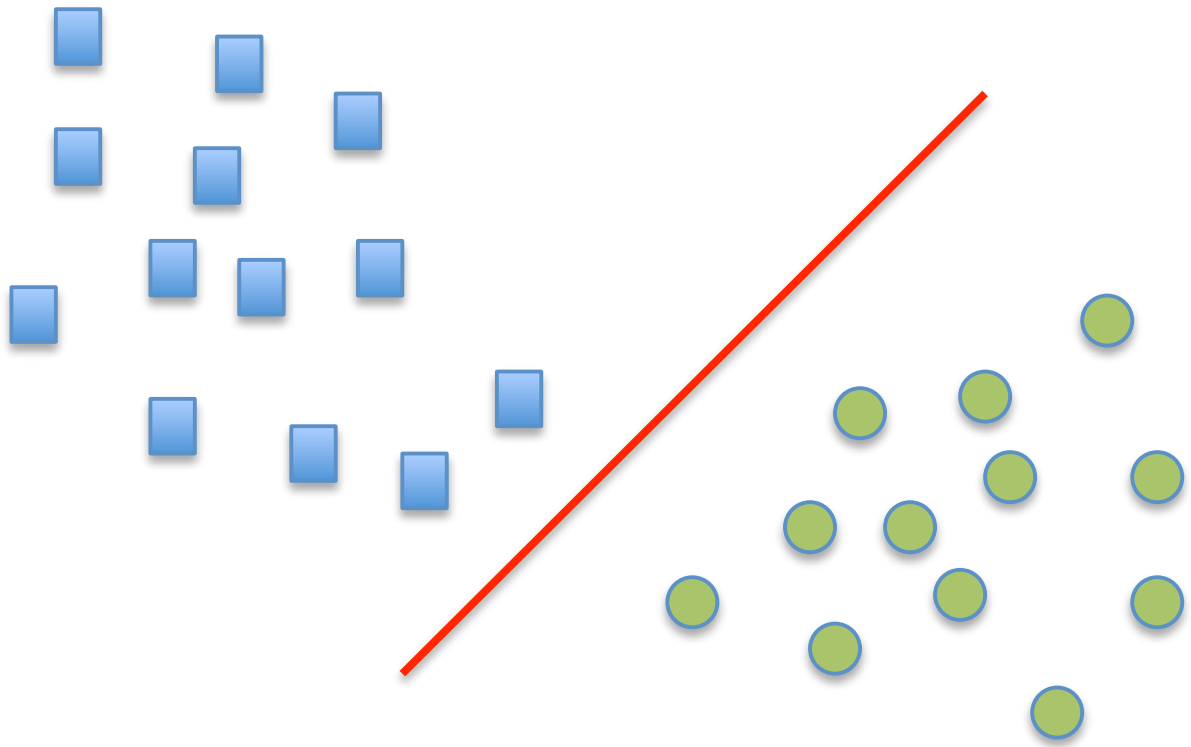


- The linear classifier has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

  which separates the space into two "half-spaces"
- In 1D this is simply a threshold

Example in 2D



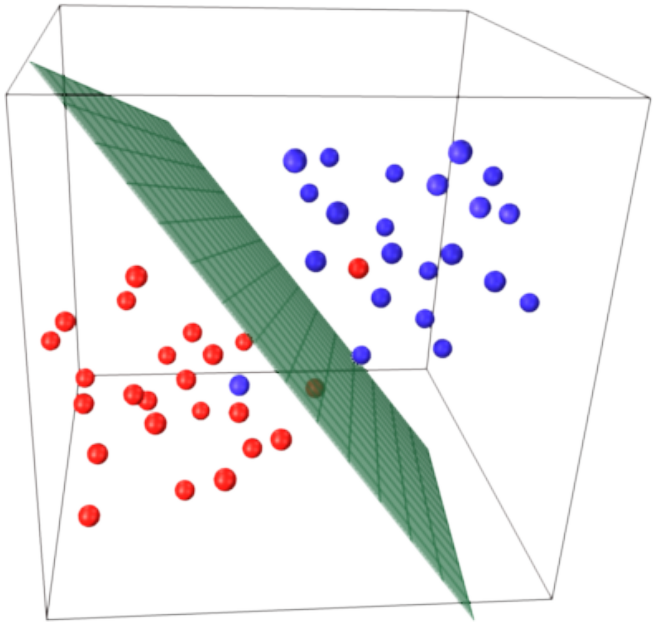
- The linear classifier has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

  which separates the space into two "half-spaces"

- In 2D this is a line

Example in 3D



- The linear classifier has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

  which separates the space into two "half-spaces"
- In 3D this is a plane
- What about higher-dimensional spaces?

Geometry

$\mathbf{w}^T\mathbf{x} = 0$ a line passing though the origin and orthogonal to $\mathbf{w}$
$\mathbf{w}^T\mathbf{x} + w_0 = 0$ shifts it by w_0

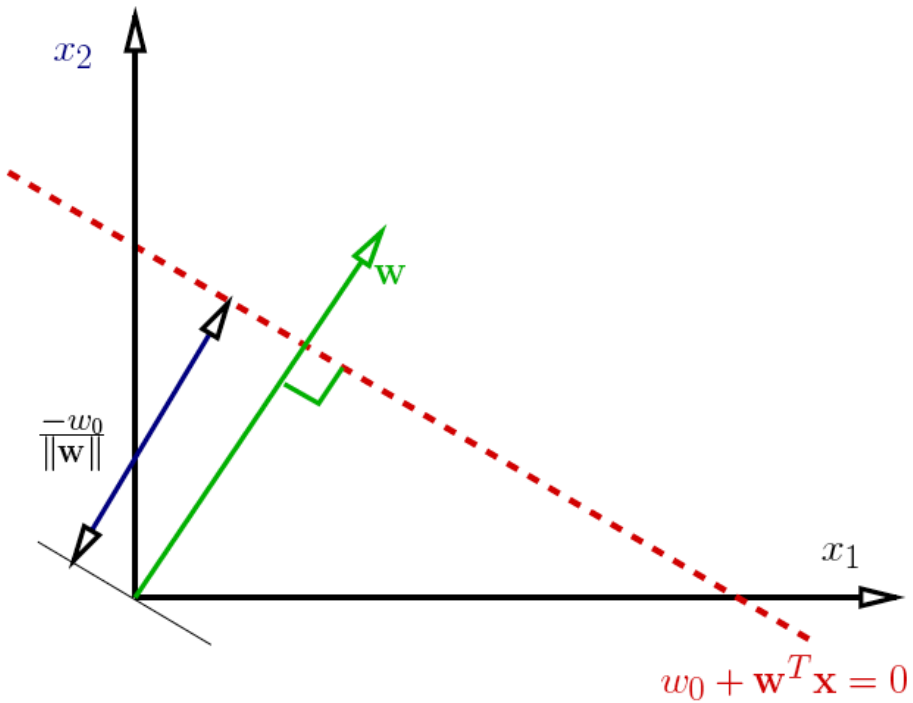
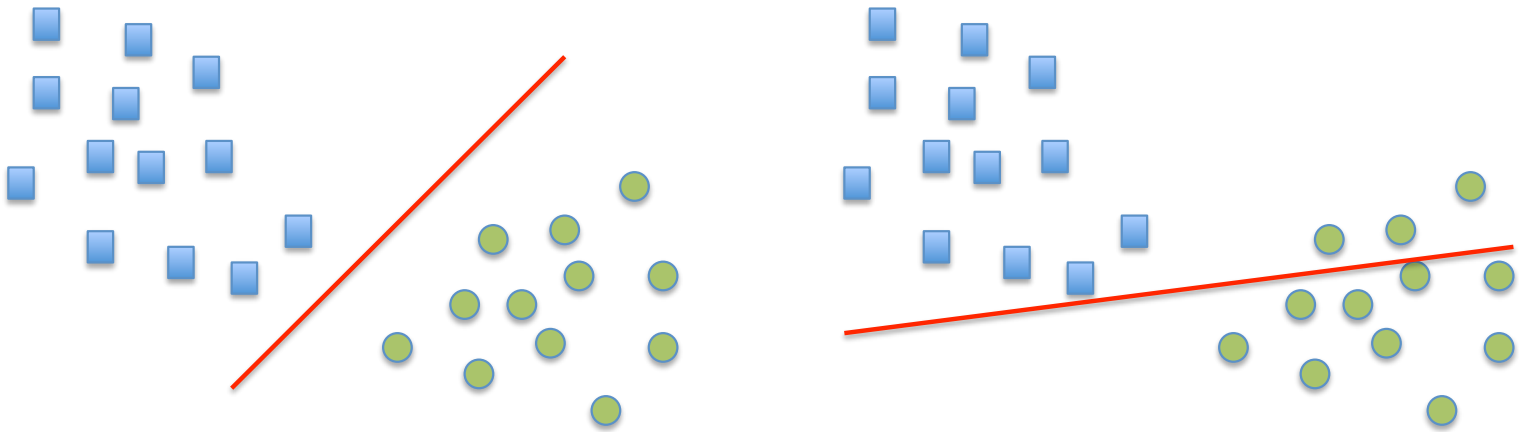


Figure from G. Shakhnarovich

Learning Linear Classifiers

- Learning consists in estimating a "good" decision boundary
- We need to find $\mathbf{w}$ (direction) and w_0 (location) of the boundary
- What does "good" mean?
- Is this boundary good?



- We need a criteria that tell us how to select the parameters
- Do you know any?

Loss functions

- Classifying using a linear decision boundary reduces the data dimension to 1

$$y(\mathbf{x}) = \text{sign}(w_0 + \mathbf{w}^T\mathbf{x})$$

- What is the cost of being wrong?
- Loss function: $L(y, t)$ is the loss incurred for predicting y when correct answer is t
- For medical diagnosis: For a diabetes screening test is it better to have false positives or false negatives?
- For movie ratings: The "truth" is that Alice thinks E.T. is worthy of a 4. How bad is it to predict a 5? How about a 2?

Loss functions

- A possible loss to minimize is the zero/one loss

$$L(y(\mathbf{x}), t) = \begin{cases} 0 & \text{if } y(\mathbf{x}) = t \\ 1 & \text{if } y(\mathbf{x}) \neq t \end{cases}$$

- Is this minimization easy to do? Why?

Other Loss functions

- Zero/one loss for a classifier

$$L_{0-1}(y(\mathbf{x}), t) = \begin{cases} 0 & \text{if } y(\mathbf{x}) = t \\ 1 & \text{if } y(\mathbf{x}) \neq t \end{cases}$$

- Asymmetric Binary Loss

$$L_{ABL}(y(\mathbf{x}), t) = \begin{cases} \alpha & \text{if } y(\mathbf{x}) = 1 \wedge t = 0 \\ \beta & \text{if } y(\mathbf{x}) = 0 \wedge t = 1 \\ 0 & \text{if } y(\mathbf{x}) = t \end{cases}$$

- Squared (quadratic) loss

$$L_{squared}(y(\mathbf{x}), t) = (t - y(\mathbf{x}))^2$$

- Absolute Error

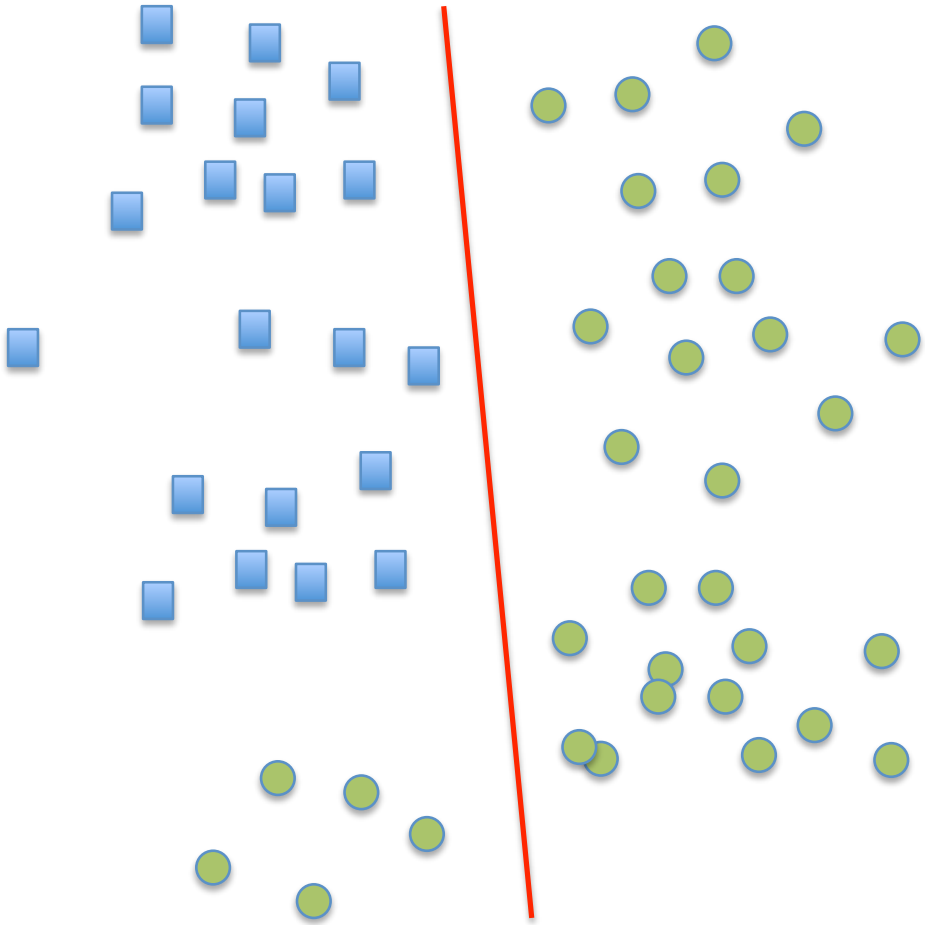
$$L_{absolute}(y(\mathbf{x}), t) = |t - y(\mathbf{x})|$$

More Complex Loss Functions

- What if the movie predictions are used for rankings? Now the predicted ratings don't matter, just the order that they imply.
- In what order does Alice prefer E.T., Amelie and Titanic?
- Possibilities:
  - 0-1 loss on the winner
  - Permutation distance
  - Accuracy of top K movies.

Can we always separate the classes?

- If we can separate the classes, the problem is linearly separable



Can we always separate the classes?

Causes of non perfect separation:

- Model is too simple
- Noise in the inputs (i.e., data attributes)
- Simple features that do not account for all variations
- Errors in data targets (miss labelings)

Should we make the model complex enough to have perfect separation in the training data?

Metrics

How to evaluate how good my classifier is? How is it doing on dog vs no-dog?

TP (True Positive) FP (False Positive) FN (False Negative)

Metrics

How to evaluate how good my classifier is?

- Recall: is the fraction of relevant instances that are retrieved

$$R = \frac{TP}{TP + FN} = \frac{TP}{\text{all groundtruth instances}}$$

- Precision: is the fraction of retrieved instances that are relevant

$$P = \frac{TP}{TP + FP} = \frac{TP}{\text{all predicted}}$$

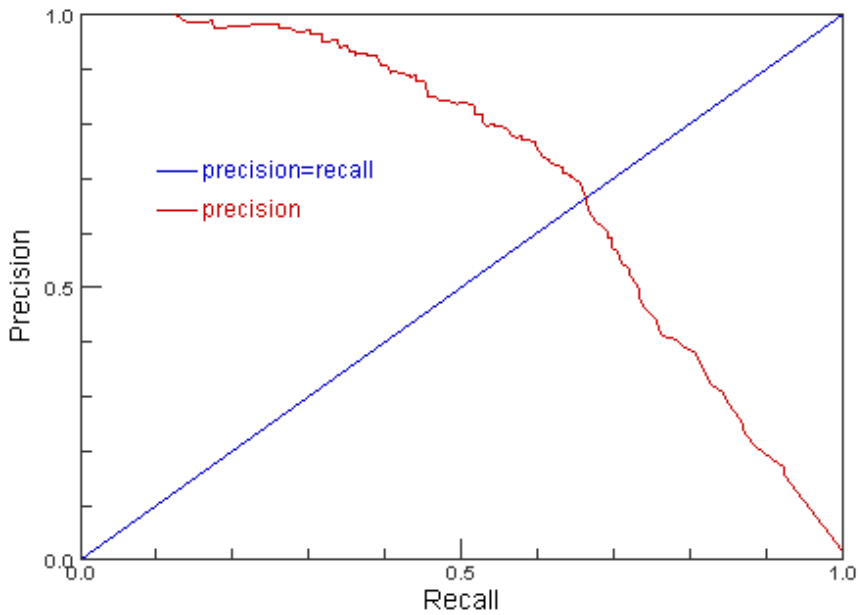
- F1 score: harmonic mean of precision and recall

$$F1 = 2\frac{P \cdot R}{P + R}$$

More on Metrics

How to evaluate how good my classifier is?

- Precision: is the fraction of retrieved instances that are relevant
- Recall: is the fraction of relevant instances that are retrieved
- Precision Recall Curve



- Average Precision (AP): mean under the curve

Metrics vs Loss

- Metrics on a dataset is what we care about (performance)
- We typically cannot directly optimize for the metrics
- Our loss function should reflect the problem we are solving. We then hope it will yield models that will do well on our dataset