Probabilistic Belief Logics

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Abstract

Modal logics based on Kripke style semantics are the prominent formalism in AI for modeling beliefs. Kripke semantics involve a collection of possible worlds and a relation among the worlds, called an accessibility relation. Dependent on the properties of the accessibility relation different modal operators can be captured. Belief operators have been modeled by an accessibility relation which produces the modal logic *KD45*. This paper demonstrates how the belief operator can also be modeled with a probability distribution over the possible worlds. It is proved that the probabilistic semantics produces the same logic. The probabilistic approach has the advantage of intuitive simplicity. Furthermore, it is demonstrated how the probability semantics can be used to construct a probability logic that is capable of representing and reasoning with a much wider variety of belief notions than the traditional modal approach.

1 Introduction

Modal logics, with possible world semantics, have proved useful for modeling notions of knowledge and belief. The basic ideas behind this approach stems from early work by Hintikka [7], who developed a possible worlds semantics for these two notions (see [6] for a useful introduction). Hintikka's logic of belief serves best as a model for the beliefs of a logically omniscience agent, i.e., an agent with unlimited computational resources who believes all logical consequences of his beliefs. The problem of "logical omniscience" has received much attention in AI, e.g., [10, 3, 8].

Although logical omniscience is still an unresolved problem, most formal work on modeling beliefs in AI continues to be done in a modal logic framework. That is, the problem of logical omniscience is not taken as a reason to reject Hintikka's approach; rather, it is taken as a reason to augment it by adding notions to model computational limitations. The modal logic approach provides a formal and analyzable mechanism for representing and

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reasoning with assertions of belief. That is, modal logics give one a declarative representation for assertions of belief like "the agent believes that the prospects for world peace are improving." Declarative representations of knowledge are independent of the uses of the knowledge, and thus are not dependent on any particular implementation technology. Furthermore, the modal logic approach provides a formal semantic interpretation for the belief assertions. The formal semantics provides an invaluable guide in analyzing legitimate modes of reasoning using the represented beliefs.

Modal logics can be given a Kripke style possible world semantics where there is a set of possible worlds and an accessibility relation between the worlds. In this paper we will explore a probabilistic approach to belief logics where we replace the accessibility relation with a probability distribution. We will show that this has the advantage of being simpler than the standard Kripke semantics, while at the same time being logically equivalent. We then demonstrate how a generalized belief logic can be constructed, based on probabilities. Such a logic has been constructed by Halpern [5] and, as we will demonstrate, it can be used to represent and reason with *graded* notions of belief, i.e., degrees of belief.

2 Modal Belief Logics

Belief logics are constructed by adding a modal belief operator, say B, to a non-modal logic. Here we will start with a *first-order* non-modal logic. The first-order language is the language in which the agent expresses his beliefs. For example, if we wish to model an agent who has beliefs about various birds and types of birds, then the first-order language may include predicate symbols like `ostrich` and `penguin`, along with constant symbols like `Tweety` and `Opus`. With this language the agent can make various assertions about his environment like $\texttt{penguin(Opus)} \land \neg\texttt{ostrich(Tweety)}$. We will call formulas like these, expressed entirely in the non-modal language, *objective* assertions. The underlying non-modal language determines the types of objective assertions that can be expressed. When we add the belief operator B to the language, we generate new formulas by applying the operator to existent formulas. The belief operator gives us a mechanism for making assertions about the agent's beliefs. For example, we can now assert that the agent believes the previous assertion with the formula $\texttt{B(penguin(Opus)} \land \neg\texttt{ostrich(Tweety))}$.

Formally, the syntax of the belief logic includes all of the formula formation rules of first-order logic along with the rule "If α is a formula, then $B\alpha$ is a formula." That is, we can prepend the modal operator B to any formula. Note that we can continue to apply first-order formula formation rules to these belief formulas. So, for example, our specification allows for formulas like $\forall\texttt{x}.B(\texttt{P(x)})$, where we have applied quantification to a belief formula. That is, our language allows arbitrary nesting of quantification and the belief operator.

Let us call the underlying first-order language $\mathcal{L}$, and the belief language formed by adding the B operator $\mathcal{L}^B$.

The aim of adding the belief modality is to allow us to capture in our formalism the intuitive behavior of beliefs. This is accomplished by prescribing properties for the B operator. Given that we are modeling the beliefs of a perfectly rational agent, we would expect the agent to have a consistent set of beliefs and for him to believe all logical consequences of his beliefs. We would also expect that the agent is capable of both positive and negative introspection. That is, if the agent believes α he should believe that he believes it, similarly if he does not believe α. Finally, and most importantly, we would not expect that what the agent believes about his environment is necessarily true of the environment: the agent may have false beliefs. It should be noted that this last condition does not conflict with our assumption that the agent is perfectly rational or logically omniscient: even though it may be reasonable to assume as a competence model that the agent has perfect reasoning capabilities it is not reasonable to assume that the agent has perfect information about the world.

These characteristics of belief are perhaps best captured by the modal logic *KD45* (a.k.a. weak $S5+$ consistency) [10, 3]. *KD45* stands for the four axioms that capture the above properties of the belief operator.

K) $(B\alpha \wedge B(\alpha \rightarrow \beta)) \rightarrow B\beta$.

D) $\neg B \mathit{false}$.

4) $B\alpha \rightarrow BB\alpha$.

5) $\neg B\alpha \rightarrow B\neg B\alpha$.

K specifies that beliefs are closed under logical consequence, D insures that a falsehood is not believed (and thus beliefs are not inconsistent), and *4* and *5* say that beliefs are closed under positive and negative introspection.

To interpret the formulas of $\mathcal{L}^B$ we can use the following denotational structure.

Definition 1 (*KD45* Modal Structures) A *KD45* structure M is a tuple of the following form.

$$M = \langle \mathcal{O}, S, R, \vartheta \rangle,$$

where $\mathcal{O}$, S, and ϑ are a domain of discourse, a set of states, and a state dependent interpretation function respectively, and R is a binary accessibility relation on S. That is, various pairs of states $\langle s_i, s_j \rangle$ are members of R. We will call the set of states s' such that $R(s, s')$ the states accessible from s.

The interpretation function ϑ determines the denotation of the predicate and function symbols. This denotation varies from state to state; therefore, a non-modal formula, like $\mathsf{P}(\mathsf{c})$ may be satisfied in some states but not in others.[1]

[1]Note that although the denotation of the symbols varies from state to state, in our model the domain of discourse does not. This has no special significance: it is simply a technical convenience. In particular, it means that the Barcan formula, $\forall x.[B\alpha] \rightarrow B[\forall x.\alpha]$, is valid for our *KD45* modal structures.

Using this semantic structure the formulas of $\mathcal{L}^B$ are interpreted with respect to a triple (M, s, v) which consists of a modal structure M, a particular state $s \in S$ (where S is the set of states in M) called the current state, and a variable assignment function that maps the variables to elements of $\mathcal{O}$. The interpretation of the formulas is determined by the the standard first-order interpretation rules augmented by a rule for interpreting belief formulas: $(M, s, v) \models B\alpha$ iff

$$(M, s', v) \models \alpha \text{ for all } s' \text{ such that } R(s, s').$$

That is, the current world satisfies a belief formula if the operand of the belief operator is satisfied by all worlds accessible from the current world.

As usual we will call a formula α of $\mathcal{L}^B$ *satisfiable* with respect to *KD45* modal structures if there exists a triple (M, s, v) such that $(M, s, v) \models \alpha$. We say that α is *valid* if for every triple (M, s, v) we have $(M, s, v) \models \alpha$. So, for example, $B\alpha \vee \neg B\alpha$ is valid, and $B\alpha \vee B\neg\alpha$ is satisfiable but not valid.

Kripke's insight was that by placing various restrictions of the accessibility relation one could capture different types of modal operators. In this case in order to insure that B obeys the axioms *K*, *D*, *4* and *5* we require three conditions on the accessibility relation:

1. R must be *serial*, i.e., for all $s \in S$ there is some s' such that $(s, s') \in R$. This conditions guarantees axiom D.

2. R must be *Euclidean*, i.e., $(s, t) \in R$ and $(s, u) \in R$ implies that $(t, u) \in R$. This condition guarantees axiom *5*.

3. R must be *transitive*, i.e., $(s, t) \in R$ and $(t, u) \in R$ implies that $(s, u) \in R$. This condition guarantees axiom *4*.

That B satisfies K follows from the fact that $B\alpha \wedge B(\alpha \rightarrow \beta)$ will be true if α and $\alpha \rightarrow \beta$ are true in all accessible worlds. This means that β will be true in all accessible worlds also.

The advantage of these three conditions on R is that they do not force R to be reflexive. Therefore the axiom $B\alpha \rightarrow \alpha$ is not valid. That is, the belief in α does not entail the truth of α: beliefs can be fallible.

3 Probabilistic Semantics

In this section we present an alternative semantics for the modal logic *KD45*. The semantics is based on a probability structure, where we have a probability distribution over the set of states instead of an accessibility relation. We will also show that the resulting logic is identical. That is, the set of formulas of $\mathcal{L}^B$ that are valid for *KD45* modal structures is exactly the set of formulas that are valid for the probability structures.

Definition 2 (Probability Structures) A probability structure P is the following tuple:

$$P = \langle \mathcal{O}, S, \mu, \vartheta \rangle$$

Here $\mathcal{O}$, S and ϑ are exactly as they were in our modal structures, and μ is a discrete probability measure on S. That is, μ is a function that maps the elements of S to the real interval [0,1] such that $\sum_{s \in S} \mu(s) = 1$. This function defines a probability distribution over the subsets of S by the following device: for every $A \subseteq S$ we define $\mu(A) = \sum_{s \in A} \mu(s)$.[2]

We can use the probability structures to interpret the formulas of $\mathcal{L}^B$ in a manner similar to the *KD45* modal structures. In particular, we use a triple (P, s, v) to interpret the formulas and except for the belief operator the rules of interpretation are identical. The new rule for the belief operator is

$$(P, s, v) \models B\alpha \qquad \text{iff} \qquad \mu\{s' : (P, s', v) \models \alpha\} = 1.$$

That is, the agent believes α if and only if the measure of the set of worlds which satisfy α is 1.

We have analogous definitions of validity and satisfiability with respect to probability structures.

To show that this new probabilistic semantics for $\mathcal{L}^B$ is logically equivalent to the previous *KD45* modal structure we have the following theorem (for the proof see Bacchus [2]).

Theorem 3 *A formula α of the language $\mathcal{L}^B$ is valid for* KD45 *modal structures if and only if it is valid for probability structures.*

In particular, this theorem implies that the axioms *K*, *D*, *4*, and *5* are all valid with respect to probability structures. Hence, the probability structures provide a model for logically closed fallible beliefs.

There have been a number of results in philosophy showing that different logics, modal and non-modal, can be given probabilistic semantics (e.g., [9, 11]), but most of these results have been for the propositional, non-quantified case. Similarly, Halpern [4] has given results on probabilistic interpretations of propositional *KD45*. To the author's knowledge this is the first result of this form for quantified *KD45*.

Intuitively multiple possible worlds model the agent's incomplete knowledge. For example, if the agent does not know the color of `Clyde` the elephant, he may think that `gray(Clyde)` is both possibly true and possibly false. The different possible worlds model the different states that would result if this assertion was alternatively true or false. The agent's belief's may be more refined than simple admission of possibility: he may feel that certain worlds are more likely than other. For example, the agent may consider `gray(Clyde)` to be more likely than ¬`gray(Clyde)`. That is, he may consider the worlds that satisfy `gray(Clyde)` to be more probable. The probability distribution in the probability structure allows us to model these *degrees* of commitment.

[2]Note that under this probability measure there are no non-measurable sets, i.e., every subset of S will have a probability no matter what the cardinality of S is. It is the case, however, that at most a denumerable number of states in S will have non-zero probability.

The language $\mathcal{L}^B$, however, is only capable of representing *full* belief. In the probabilistic interpretation a belief formula like $B\alpha$ denotes that the probability of the set of worlds that satisfy α is 1. This interpretation of B is fixed, and there is no way for us to refer to intermediate degrees of belief. That is, the language is not sufficiently expressive to represent intermediate degrees of belief. It is instructive, however, to examine the probabilistic interpretation of full belief.

The *KD45* modal structures, and *KD45* modal logic, models full belief. That is, although the agent's beliefs may be false the agent does not consider this to be a serious possibility. The agent's beliefs are not falsified by any accessible world. It is for this reason that the agent is willing to believe all deductive consequences of his beliefs. This situation is similar in the probabilistic interpretation. In the agent's view if he believes α, then he considers the probability that α is false to be zero. It is logically possible that α could be false, but the worlds that falsify it all have probability zero.

4 Probability Logics

It can be argued that our probability semantics provides a simpler more transparent semantics for beliefs. Instead of a complex accessibility relation we have a simpler probability distribution. Furthermore, there is a plausible intuitive basis for the probability distribution: the agent may consider certain worlds to be more likely than others. There does not seem to be any such direct intuitive basis for the accessibility relation. Rather the suitability of the accessibility relation is determined primarily by how well it captures our intuitions about the behavior of the belief operator.

Beyond this advantage, however, if we are dealing just with the language $\mathcal{L}^B$ there does not seems to be much need for a probabilistic semantics: $\mathcal{L}^B$ is not sufficiently expressive to use the full power of the semantics. Given a formula α the probabilistic semantics allows one to evaluate the measure of the set of worlds which satisfy α. This can be intuitively be regarded as being the agent's degree of belief in α. There is no reason for this degree of belief to be 1; it can be any value in the unit interval. By using the full power of the probabilistic semantics we can capture a notion of graded belief and represent and reason with many complex assertions about belief beyond full belief.

The key to accomplishing this is to extend the expressive power of the language $\mathcal{L}^B$ so that it can make direct reference to the probabilities. In this manner we can make various *qualitative* and quantitative assertions about the probabilities, and we can reason *with the probabilities.* Since these probabilities are to be interpreted as degrees of belief this would enable us to represent and reason with the agent's degrees of beliefs. A method for making direct reference to probabilities in a logical language has been developed by Bacchus [1]. This method was adopted and refined by Halpern who has developed an expressive probability logic based on the probability structures that we have used above. We will use a syntactic variant of Halpern's logic (to be precise Halpern's Type II probability logic) [5]. This is the variant that is used in Bacchus [2] which contains an extensive exposition of the logic, including many examples.

We will call this more expressive language $\mathcal{L}^{\text{prob}}$. It is a two-sorted language. One of the sorts consists of the objects in the agent's environment, i.e., the solitary sort that was used in $\mathcal{L}^B$. The second, new sort is a numeric sort. The terms of the numeric sort will denote real numbers, and the numeric functions and predicates will be functions and relations over the reals. The complete language consists of a set of object functions and predicates, and a set of numeric functions and predicates. Among the numeric functions and predicates are the symbols 1, 0, -1, $+$, $\times$, $=$, $<$. The numeric functions and predicates take only numeric terms are arguments, similarly for the object functions and predicates. That is, there are no mixed functions or predicates. The formulas of $\mathcal{L}^{\text{prob}}$ are formed by the standard first-order rules of formation, with the addition of a rule for generating probability terms:

If α is a formula then $\mathsf{prob}(\alpha)$ is a *numeric term.*

Semantically, the language is interpreted with respect to a triple (P, s, v) consisting of a probability structure, a current world, and a variable assignment function. The rules of interpretation are standard except for the rule that interprets the probability terms:

$$\mathsf{prob}(\alpha)^{(P,s,v)} = \mu\{s' : (P, s', v) \models \alpha\}$$

That is, the denotation of the probability term $\mathsf{prob}(\alpha)$ at any world s is the probability of the set of worlds that satisfy α.

It is easy to see that the denotation of $\mathsf{prob}(\alpha)$ is independent of the current worlds s, and that since probabilities are real numbers it denotes a real number; i.e., it is a legitimate numeric term.

The essential difference between $\mathcal{L}^B$ and $\mathcal{L}^{\text{prob}}$ is the replacement of the belief operator B with a probability operator prob. Both operators take formulas as their arguments, but the belief operator yields a new formula while the probability operator yields a new *term.* This is how the increased expressiveness of $\mathcal{L}^{\text{prob}}$ is accomplished. $B\alpha$ is a formula of $\mathcal{L}^B$ with a truth value given by a fixed rule of interpretation; on the other hand $\mathsf{prob}(\alpha)$ is not a formula of $\mathcal{L}^{\text{prob}}$, it is a term which denotes a real number: the probability of the set of worlds that satisfy α. In order to produce a formula of $\mathcal{L}^{\text{prob}}$ we have to use $\mathsf{prob}(\alpha)$ in a formula; in particular we have to use it as an argument to a predicate. For example, we can use it as an argument to the numeric equality predicate along with another numeric term, say 1. This would produce the formula $\mathsf{prob}(\alpha) = 1$. As theorem 3 demonstrates, this formula is logically equivalent to $B\alpha$. We could chose another numeric term instead of 1; for example, the formula $\mathsf{prob}(\alpha) = \mathsf{prob}(\beta)$ says that the agent has equal degree of belief in both α and β, without making any commitment to the exact values of this degree of belief.

We now present some examples of what can be represented in $\mathcal{L}^{\text{prob}}$.

5 Examples of Expressiveness

Example 1 (Non-extreme commitment) We could represent the agent's belief that John probably has some type of cancer.

$$\mathsf{prob}(\exists \mathtt{x}.\mathtt{has\_cancer\_type}(\mathtt{John}, \mathtt{x})) > 0.5^3.$$

Here we assume that the domain of objects includes a set of individuals which are types of cancer, and in each of the satisfying worlds one of these individuals lies in the `has_cancer_type` relation with the individual denoted by the constant John. It should be noted that this formula makes no commitment about the particular type of cancer that John has: there could be a different satisfying `x` in each world. Hence, we are not forced to over-commit in our representation of the agent's beliefs. Finally, we can note that this formula is satisfied by many different structures, each of which may have a different probability distribution over the states. That is, the actual probability measure of the set of satisfying states of this formula could be any number, as long as it is greater than 0.5. We do not have to know the agent's probability distribution over the set of worlds (it is highly unlikely that the agent himself knows this). Rather, as with most logical formalisms, we can represent what we do know of the agent's beliefs and can reason about the constraints that these beliefs place on other beliefs.

Example 2 (Relative information about beliefs) Perhaps the agent believes that it is more likely that John has lung cancer than any other type of cancer.

$$\begin{aligned} &\forall \mathtt{x}.\mathtt{cancer\_type}(\mathtt{x}) \wedge [\mathtt{x} \neq \mathtt{lung}] \\ &\quad \rightarrow \mathsf{prob}(\mathtt{has\_cancer\_type}(\mathtt{John}, \mathtt{lung})) \\ &\qquad > \mathsf{prob}(\mathtt{has\_cancer\_type}(\mathtt{John}, \mathtt{x})). \end{aligned}$$

Note that in this example the quantification of `x` occurs outside of the probability context. So by the time we interpret the probability operator the variable `x` has already been given a particular assignment. Furthermore, the outermost universal quantification runs through all possible assignments to `x`. All those `x` which satisfy the antecedent of the implication must satisfy the probabilistic constraint in the consequent.

We may have more complex arithmetic relations between beliefs. For example, the agent may believe that it is more than twice as likely that John has skin cancer than lung cancer.

$$\begin{aligned} &\mathsf{prob}(\mathtt{has\_cancer\_type}(\mathtt{John}, \mathtt{skin})) \\ &\quad > 2 \times \mathsf{prob}(\mathtt{has\_cancer\_type}(\mathtt{John}, \mathtt{lung})). \end{aligned}$$

The agent may have more quantitative beliefs, e.g., the probability that John has cancer lies in the interval 0.6 to 0.95.

$$\mathsf{prob}(\exists \mathtt{x}.\mathtt{has\_cancer\_type}(\mathtt{John}, \mathtt{x})) \in [0.6, 0.95].$$

[3]Extra numeric constants like 0.5 can be added to $\mathcal{L}^{\mathrm{prob}}$ by definition. For example, we can define 0.5 by the formula '$0.5 \times (1+1) = 1$.' That is, where ever 0.5 occurs we can use equality to remove the new constant returning to the basic $\mathcal{L}^{\mathrm{prob}}$ language

6 Reasoning

It is possible to give a proof theory for $\mathcal{L}^{\text{prob}}$. The proof theory consists of the first-order axioms, axioms for reasoning about real numbers,[4] and probability axioms. The proof theory is powerful enough to do an extensive amount of probabilistic and first-order reasoning, and is complete for certain special cases (see Halpern [5]). Bacchus [2] contains some detailed examples of how one can use the proof theory to perform various types of reasoning.

The proof theory is powerful enough to fully capture the logic of full belief. That is, it can deduce all formulas that are valid for *KD45* modal logic. As we have seen, the full belief formulas of *KD45* are logically equivalent to formulas with probability one, i.e., $B\alpha \equiv \mathsf{prob}(\alpha) = 1$. Hence there is a natural translation from the formulas of $\mathcal{L}^B$ into $\mathcal{L}^{\text{prob}}$: just replace every subformula of the form $B(\alpha)$ by a subformula of the form $\mathsf{prob}(\alpha) = 1$. It can be demonstrated that the $\mathcal{L}^{\text{prob}}$ translation of every valid $\mathcal{L}^B$ formula is deducible from $\mathcal{L}^{\text{prob}}$'s proof theory.

Besides logical reasoning with full belief, the proof theory can perform an extensive amount of probabilistic reasoning. For example, it can perform the probabilistic reasoning involved in Bayes' networks (Pearl [12]). Both of these claims are demonstrated in [2].

7 Conclusions

We have demonstrated that a probabilistic interpretation provides a generalization of modal approaches to belief. In particular, we have proved that probability logics can provide a proper generalization of quantified *KD45*. We have also demonstrated how such logics can be used to represent and reason with graded notions of belief.

Another application of probabilities is to provide a logic capable of representing statistical assertions. This use of probabilities is to be distinguished from our use of probabilities here to model degrees of belief. Statistical assertions are assertions about the state of the agent's environment just like logical assertions such as `gray(Clyde)`; they are not assertions about the agent's beliefs. To express statistical assertions a more expressive language has been developed (Bacchus [1]). We can call this language $\mathcal{L}^{\text{stat}}$. By using the more expressive $\mathcal{L}^{\text{stat}}$ as the underlying non-modal language instead of $\mathcal{L}$, we can develop a probabilistic belief logic capable of expressing an agent's logical and statistical beliefs. This belief logic has applications in non-monotonic reasoning [2].

In summary, one of the objectives of this paper is to demonstrate that probabilities and logic can co-exist comfortably in a single formalism, and that such a combination can often yield a more powerful tool than just logic or just probabilities. We have demonstrated this for the case of belief logics. The above mentioned application of statistical probabilities to non-monotonic reasoning is another example of this synergism.

[4]These are the axioms of real closed fields (Tarski [13]). They capture the algebraic behavior of the reals.

References

[1] Fahiem Bacchus. *Representing and Reasoning With Probabilistic Knowledge.* PhD thesis, The University of Alberta, 1988.

[2] Fahiem Bacchus. *Representing and Reasoning With Probabilistic Knowledge.* MIT-Press, Cambridge, Massachusetts, 1990.

[3] Ronald Fagin and Joseph Y. Halpern. Belief, awareness and limited reasoning. *Artificial Intelligence*, 34:39–76, 1988.

[4] Joseph Y. Halpern. The relationship between knowledge, belief, and certainty: Preliminary report. In *Proceedings of the Fifth Workshop on Uncertainty in Artificial Intelligence*, pages 143–151, 1989.

[5] Joseph Y. Halpern. An analysis of first-order logics of probability. *Artificial Intelligence*, 46:311–350, 1990.

[6] Joseph Y. Halpern and Yorham Moses. A guide to the modal logics of knowledge and belief. In *Procceedings of the International Joint Conference on Artifical Intelligence (IJCAI)*, pages 480–490, 1985.

[7] Jaakko Hintikka. *Knowledge and Belief.* Cornell University Press, Ithaca, 1962.

[8] Kurt Konolige. *A Deduction Model of Belief.* Research Notes in Artificial Intelligence. Morgan Kaufmann, San Mateo, California, 1986.

[9] Hughes LeBlanc. Alternatives to standard first-order semantics. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic. Vol II*, pages 225–258. D. Reidel, Dordrecht, Netherlands, 1983.

[10] Hector J. Levesque. A logic of implicit and explicit belief. In *Proceedings of the AAAI National Conference*, pages 198–202, 1984.

[11] Charles G. Morgan. Simple probabilistic semantics for propositional K, T, B, S4 and S5. *Journal of Philosophic Logic*, 11:443–458, 1982.

[12] Judea Pearl. *Probabilistic Reasoning in Intelligent Systems.* Morgan Kaufmann, San Mateo, California, 1988.

[13] A. Tarski. *A Decision Method for Elementary Algebra and Geometry, 2nd Edition.* University of California Press, Berkeley, 1951.