TS Method – Summary

Let $T_k(x, y_{j-1})$ denote the first $k+1$ terms of the Taylor series expanded about the discrete approximation, (x_{j-1}, y_{j-1}), and $\hat{z}_{k,j}(x)$ be the polynomial approximation (to $y(x)$) associated with this truncated Taylor series,

$$\begin{aligned} \hat{z}_{k,j}(x) &= y_{j-1} + \Delta\, T_k(x, y_{j-1}), \\ T_k(x, y_{j-1}) &\equiv f(x_{j-1}, y_{j-1}) + \frac{\Delta}{2} f^{'}(x_{j-1}, y_{j-1}) \cdots + \frac{\Delta^{k-1}}{k!} f^{(k-1)}(x_{j-1}, y_{j-1}), \end{aligned}$$

where $\Delta = (x - x_{j-1})$.

A simple, constant stepsize (fixed h) TS method is then given by:

-Set $h = (b-a)/N$;

-for $j = 1, 2, \cdots N$

$x_j = x_{j-1} + h$;

$y_j = y_{j-1} + h\, T_k(x_j, y_{j-1})$;

-end

Local/Global Errors

Note that, strictly speaking, $z_{k,j}(x)$ is not a direct approximation to $y(x)$ but to the solution of the 'local' IVP:

$$z_j' = f(x, z_j), \quad z_j(x_{j-1}) = y_{j-1}.$$

Since y_{j-1} will not be equal to $y(x_{j-1})$ in general, the solution to this local problem, $z_j(x)$, will not then be the same as $y(x)$.

To understand and appreciate the implications of this observation we distinguish between the 'local' and 'global' errors.

Definitions:

- The local error associated with step j is $z_j(x_j) - y_j$.
- The global error at x_j is $y(x_j) - y_j$.

A Classical Approach

A Classical (pre 1965) numerical method approximates $y(x)$ by dividing $[a, b]$ into equally spaced subintervals, $x_j = a + j\ h$ (where $h = (b - a)/N$) and, proceeding in a step-by-step fashion, generates y_j after $y_1, y_2, \cdots y_{j-1}$ have been determined.

- If the Taylor series method is used in this way, then the TS theorem with remainder shows that the local error on step j (for the TS method of order k) is:

$$E_j = \frac{h^{k+1} f^{(k)}(\eta_j, z_j(\eta_j))}{(k+1))!} = \frac{h^{k+1} z_j^{(k+1)}(\eta_j)}{(k+1)!}.$$

- If $k = 1$ we have Eulers Method where $y_j = y_{j-1} + h\ f(x_{j-1}, y_{j-1})$, and the associated local error satisfies,

$$LE_j = \frac{h^2}{2} y''(\eta_j).$$

Error Bounds for IVP Methods

Definition: A method is said to converge iff,

$$\lim_{h\to 0,(N\to\infty)} \left\{ \max_{j=1,2,\cdots N} |y(x_j) - y_j| \right\} \to 0.$$

- **Theorem:** (typical of classical convergence results)
  Let $[x_j, y_j]_{j=0}^{N}$ be the approximate solution of the IVP, $y^{'} = f(x,y),\ y(a) = y_0$ over $[a,b]$ generated by Euler's method with constant stepsize $h = (b-a)/N$. If the exact solution, $y(x), \in C^2[a,b]$ and $|f_y| < L,\ |y^{''}(x)| < Y$ then the associated GE, $e_j = y(x_j) - y_j$, $x_j = a + j\ h$ satisfies (for all $j > 0$),

$$\begin{aligned} |e_j| &\le \frac{hY}{2L}(e^{(x_j - x_0)L} - 1) + e^{(x_j-x_0)L}|e_0|, \\ &\le \frac{hY}{2L}(e^{(b-a)L} - 1) + e^{(b-a)L}|e_0|. \end{aligned}$$

Observations re Convergence

1. e_0 will usually be equal to zero.
2. This bound is generally pessimistic as it is exponential in $(b - a)$ where linear error growth is often observed on practical or realistic problems.
3. In the general case one can show that when local error is $O(h^{p+1})$ the global error is $O(h^p)$.

Proof of Conv Th (outline)

Eulers Method satisfies,

$$y_j = y_{j-1} + hf(x_{j-1}, y_{j-1}).$$

A Taylor series expansion of $y(x)$ about $x = x_{j-1}$ implies

$$y(x_j) = y(x_{j-1}) + hf(x_{j-1}, y(x_{j-1})) + \frac{h^2}{2} y^{''}(\eta_j).$$

Subtracting the first equation from the second we obtain,

$$y(x_j) - y_j \;=\; y(x_{j-1}) - y_{j-1} + h[f(x_{j-1}, y(x_{j-1})) - f(x_{j-1}, y_{j-1})] + \frac{h^2}{2} y^{''}(\eta_j).$$

If $Y = \max_{x \in [a,b]} |y^{''}(x)|$ and $|f_y| \leq L$, then, from the definition of e_j and the observation that $f(x, y)$ satisfies a Lipschitz condition with respect to y, we have ...

Proof (cont)

$$
\begin{aligned}
|e_j| &\leq |e_{j-1}| + hL|y(x_{j-1}) - y_{j-1}| + |\frac{h^2}{2} y^{''}(\eta_j)|, \\
&\leq |e_{j-1}| + hL|e_{j-1}| + \frac{h^2}{2} Y, \\
&= |e_{j-1}|(1 + hL) + \frac{h^2}{2} Y.
\end{aligned}
$$

This is a linear recurrence relation (or inequality) which after some work (straightforward) can be shown to imply our desired result,

$$
|e_j| \leq \frac{hY}{2L} (e^{(b-a)L} - 1) + e^{(b-a)L} |e_0|.
$$

Note that this is only an upper bound on the global error and it may not be sharp.

An Example

Consider the following equation,

$$y' = y, \quad y(0) = 1, \quad \text{on } [0, 1].$$

Now since $\frac{\partial f}{\partial y} = 1$, $L = 1$ and since $y(x) = e^x$, we have $Y = e$ and $e_0 = 0$. Applying our error bound with $h = 1/N$ and $y_N \approx y(1) = e$ we obtain,

$$|GE_N| = |y_N - e| \leq \frac{he}{2}(e - 1) < 2.4h.$$

But for $h = .1$ we observe that $y_{10} = 2.5937..$ with an associated true error of $.1246..$ ($\equiv e - y_{10}$). This error bound is $.24$. This is an overestimate by a factor of 2.

Exercise: Compare the bound to the true error for $h = .01, \quad h = .001$.

Limitations of Classical Approach

- Analysis is valid only in the limit as $h \to 0$.
- Bounds are usually very pessimistic (can overestimate the error by several orders of magnitude).
- Analysis does not consider the affect of f.p. arithmetic.

Affect of FP Arith

Assume $fl(f(x_{j-1}, y_{j-1})) = f(x_{j-1}, y_{j-1}) + \epsilon_j$ and

$$\begin{aligned} y_j &= y_{j-1} \oplus h \otimes fl(f(x_{j-1}, y_{j-1})), \\ &= y_{j-1} + hf(x_{j-1}, y_{j-1}) + h\epsilon_j + \rho_j, \end{aligned}$$

where $|\epsilon_j|, \ |\rho_j| \ < \ \mu$.

Then, proceeding as before we obtain,

$$|e_j| < |e_{j-1}|(1 + hL) + \frac{h^2}{2}\bar{M},$$

where $\bar{M} = Y + \mu/h + \mu/(h^2)$.

Affect of FP Arith (cont)

Therefore the revised error bound becomes:

$$
\begin{aligned}
|e_j| &\leq e^{(b-a)L}|e_0| + \frac{h\bar{M}}{2L}(e^{(b-a)L} - 1), \\
&= e^{(b-a)L}|e_0| + (e^{(b-a)L} - 1)(\frac{hY}{2L} + \frac{\mu}{2L} + \frac{\mu}{2hL}).
\end{aligned}
$$

So, as $h \to 0$, the term $\frac{\mu}{2hL}$ will become unbounded (unless the precision changes) and we will not observe convergence.