CSCD37H

Assignment 3

February 25, 2013 University of Toronto **Due:** March 11, 2013

1. Scientists have recently discovered a stable orbit that arises in the simulation of the restricted three-body problem (where the orbits are planar). The bodies have equal mass (in our example we assume $m_1 = m_2 = m_3 = 1.0$) and, with the appropriate starting conditions, will follow the same figure-eight orbit as a periodic steady-state solution. Figure 1a shows the solution behaviour over several orbits by displaying the first spatial coordinate (as a function of x) over the interval of interest for each body. Figure 1b displays the "phase portrait" of the solution to this problem. This phase portrait is defined to be a plot of one spatial coordinate vs the other spatial coordinate for each of the three bodies. For the set of initial conditions in our example the solution vector is periodic and the orbits are identical (although out of phase) for each body. Note that a solution to the system of IVPs, $y' = f(x, y)$ is periodic with period T if T is the smallest positive value such that $y(x+T) = y(x)$. (ie. all components of $y(x)$ must satisfy this equality.

   The two spatial coordinates of the j^{th} body are y_{1j}, y_{2j} for $j = 1, 2, 3$. Each of the six coordinates satisfy a second-order differential equation (which can be easily derived from Newtons law of motion applied to gravitational systems – see for example, Shamine[1994, pp. 90-95]):

   $$y''_{ij} = \sum_{k=1, k\neq j}^{3} m_k \left(\frac{y_{ik} - y_{ij}}{d_{jk}^3}\right),$$

   where

   $$d_{kj}^2 = \sum_{i=1}^{2} (y_{ik} - y_{ij})^2, \ k, j = 1, 2, 3.$$

   When this system is re-written, as a first order system, the dimension of the problem is 12 and the initial conditions, at $x = 0$, are given by,

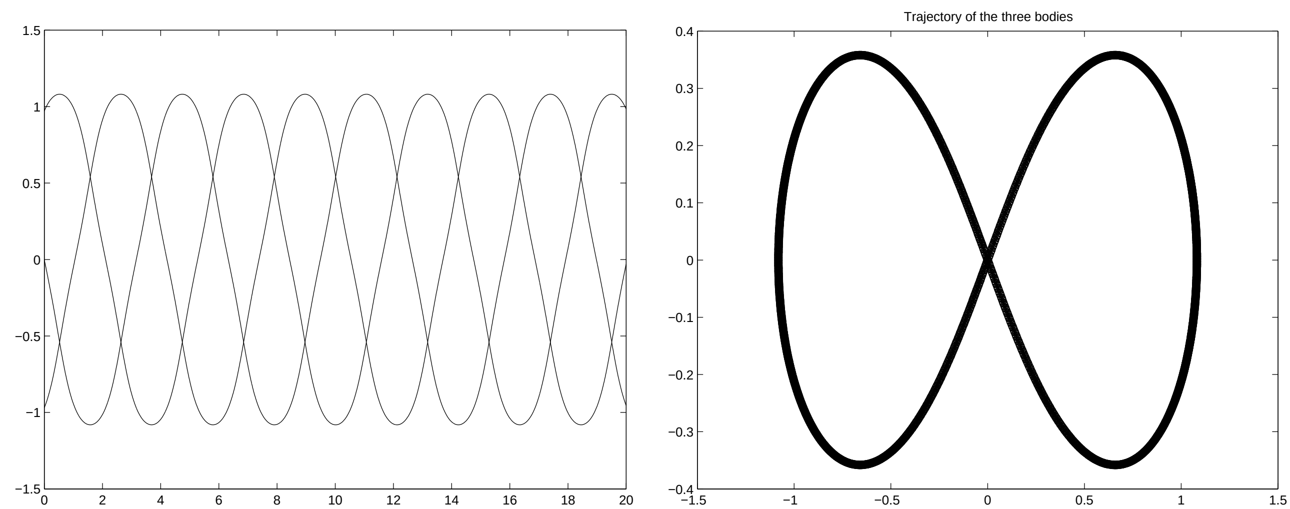
   $$\begin{aligned} y_{11} &= -0.97000436, & y'_{11} &= 0.466203685, \\ y_{21} &= 0.24308753, & y'_{21} &= 0.43236573, \\ y_{12} &= 0.0, & y'_{12} &= -0.93240737, \\ y_{22} &= 0.0, & y'_{22} &= -0.86473146, \\ y_{13} &= 0.97000436, & y'_{13} &= 0.466203685, \\ y_{23} &= -0.24308753, & y'_{23} &= 0.43236573. \end{aligned}$$

   We are interested in this solution for $x \in [0, 20]$.

   (a) For a general system of IVPs, $y' = f(x, y), y(0) = y_0$, discuss how you could determine (using the routines of Matlab) whether the solution is periodic or not.

(b) Use ode45 of MatLab to compute the solution to the above three-body problem with the given initial conditions and determine the period of the solution. Note that in order to carry out this numerical investigation you will have to make use of some of the options available with ode45 which are invoked using the routine odeset.

(c) You are to determine, through some numerical experiments, whether the the stable periodic solution observed for the three body problem with $m_1 = m_2 = m_3 = 1.0$ is sensitive to the value used for the constant mass. In particular, does the solution with the above initial conditions appear to converge to a periodic solution when $m_1 = m_2 = m_3 = M$? (and M is close to 1.0.)



(a) Solution Plot

(b) Phase Portrait

Figure 1: The approximate solution produced by ode45 for the Three Body problem

To help you get started, the MatLab script for specifying the vector of initial conditions and for evaluating the equivalent ode (after conversion to a first order system) is given by:

```
% specify the initial conditions Y0
   x00 = [-0.97000436; 0.24308753]; xp0 = [0.466203685;0.43236573];
   x10 = [0;0];                     xp1 = [-0.93240737;-0.86473146];
   x20 = [ 0.97000436;-0.24308753]; xp2 = [0.466203685;0.43236573];
   i0=[x00; xp0; x10; xp1; x20; xp2];
%---------The Function f.m (for three body problem-------------------
function Ydot = bodyf(t,Y,p1)
m0=1.0; m1=1.0; m2=1.0;   %masses of the three bodies
x0 = Y(1:2); x1 = Y(5:6); x2 = Y(9:10);
d0 = (x2-x1)/norm(x2-x1)^3;
d1 = (x0-x2)/norm(x0-x2)^3;
d2 = (x1-x0)/norm(x1-x0)^3;
Ydot(1:2) = Y(3:4);
Ydot(5:6) = Y(7:8);
```

```
Ydot(9:10) = Y(11:12);
Ydot(3:4) = m1*d2 - m2*d1;
Ydot(7:8) = m2*d0 - m0*d2;
Ydot(11:12) = m0*d1 - m1*d0;
Ydot=Ydot';
%----------------------------------------------------
```