CS1 2011 Final Exam

1. a) Any higher order ≥ 2 method is acceptable
Eg - Composite Trapezoidal has order 2 ($O(h^2)$)
(5) - Composite Simpson rule order 4 ($O(h^4)$)

b) $O(h)$ in both cases
- can be expressed by finding $\bar{x}_i \ni f(\bar{x}_i) = 0$
(10)
and one case $\bar{x}_{i-1}$ $\bar{x}_i$ are solved for (a)

c) $err = \sum_{\substack{\text{subintervals } i=1 \\ \bar{x}_{i-1}, \bar{x}_i}}^{K} |err_i|$
(5)

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a)

(5) $z = \begin{bmatrix} y \\ y' \\ u \end{bmatrix} \Rightarrow z' = \begin{bmatrix} y'(x) \\ y''(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} z_2(x) \\ z_1^3(x) + z_3^3(x) \\ z_1(x) + z_3(x) \end{bmatrix} = f(x, z), \quad z_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

1) one solution introduces $z_4(t) = \int_0^x (y'(s) + u'(s))^2 \, ds \equiv h(t)$

then $z_4'(x) = (y'(x) + u'(x))^2$

$= [z_2(x) + (z_1(x) + z_3(x))]^2$

(15) with $z_4(0) = 0$

$\therefore \; \bar{z}(x) \in \mathbb{R}^4 \equiv \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$ satisfies the IVP

$$\bar{z}'(x) = \begin{bmatrix} \bar{z}_2(x) \\ \bar{z}_1^3(x) + \bar{z}_3^3(x) \\ \bar{z}_1(x) + \bar{z}_3(x) \\ (\bar{z}_2(x) + \bar{z}_1(x) + \bar{z}_3(x))^2 \end{bmatrix} = \hat{f}(x, \bar{z}), \quad \bar{z}(0) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

can apply any ODE solver and plot $\bar{z}_4(x)$

4) a) $k_1 = f(x_0 y_0) = \lambda y_0$

(5) $k_2 = f(x_0 + h, y_0 + hk_1) = \lambda[y_0 + h\lambda y_0] = \lambda(1 + h\lambda) y_0$

b) induction hypothesis $k_r = \lambda p_{r-1}(h\lambda) y_0$ for $r = 1, 2 \cdots$

true for base case $r = 1$ since as in a)

k_1 always $= \lambda y_0 = \lambda p_0(h\lambda) y_0$

where p_0 is a polynomial of degree $0 \equiv 1$

assume for $r > 1$ $r \le n$ and ind hyp is satisfied

10 for $i < r$, ie for $i < r$, $k_i = \lambda p_{i-1}(h\lambda) y_0$

then $k_r = f(x_0 + \alpha_r h, y_0 + h \sum_{j=1}^{r-1} \beta_{rj} k_j) = \lambda(y_0 + h \sum_{j=1}^{r-1} k_j)$

but for $j < r$ $k_j = \lambda p_{j-1}(h\lambda) y_0$

so $k_r = \lambda(y_0 + h \sum_{j=1}^{r-1} \lambda p_{j-1}(h\lambda) y_0)$

$= \lambda \underbrace{\left(1 + \sum_{j=1}^{r-1} (h\lambda) p_{j-1}(h\lambda)\right)} y_0$

$= \lambda \quad p_j(h\lambda) \quad y_0$

QED

(5) c) True local solution $y(x_1) = e^{h\lambda} y_0$

$y_1 = y_0 + h \sum_{r=1}^{\hat{s}} \omega_r k_r = y_0 + \sum_{r=1} (h\lambda) p_{r-1}(h\lambda) y_0$

$= (1 + q_s(h\lambda)) y_0 = \underset{s}{P}(h\lambda) y_0$

$LE = O(h^{p+1}) = (e^{h\lambda} - \underset{s}{P}(h\lambda)) = O(h^{s+1})$

$\therefore LE = \left(e^{h\lambda} - p_s(h\lambda)\right) y_0 = O(h^{s+1})$

which means that $p_s(h\lambda)$ must

equal $\sum_{j=0}^{s} \frac{(h\lambda)^j}{j!}$!

Note this question is not easy and

I did not expect anyone to give a

complete solution. W. E.