

TWPBVPC Package: solving test problems

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1 Introduction

`twpbvpc` is a MATLAB program that calculates approximate solutions two point Boundary Value Problems. These problems may or may not be singularly perturbed. The BVPs are of the general form:

$$y' = f(x, y), \quad a \leq x \leq b \quad (1)$$

$$g(y(a), y(b)) = 0, \quad (2)$$

where $g = (g_1, g_2, \dots, g_n)^T$ is a vector functions. The functions f and g_i are assumed to be differentiable. The problem must be posed as a first-order system. This requirement is not unduly restrictive, however, since standard techniques can be used to convert an n th-order equation to a system of n first-order equations. For example, the second-order singularly perturbed problem

$$\lambda y'' = f(x, y, y'), \quad 0 < x < 1, \quad y(0) = \alpha, \quad y(1) = \beta, \quad (3)$$

where $0 < \lambda \ll 1$, $x \in \mathbb{R}$, can be converted to the following first-order system:

$$\begin{aligned} u' &= z, \\ z' &= \frac{1}{\lambda} f(x, u, z), \\ u(0) &= \alpha, u(1) = \beta. \end{aligned}$$

2 Format of the problem codes

The eight functions that define the problem are called `prob`, `fun`, `bcfun`, `dfun`, `dbcfun`, `settolerances`, `setoutput` and `esolu`. The following subsections describe the format of these subroutines in full detail.

2.1 The function prob

This routine gives some general information about the test problem.

```
function [problem,type,m,Linear,numjac,numbcjac,Vectorized,JVectorized,solinit]=prob()
```

Meaning of the arguments:

problem

This character string contains the name of the problem, e.g. `bvpT1`, and corresponds to the name of the MATLAB source file.

type

This character string takes the value `ODEBVP`.

m

The dimension of the problem, which is the number of equations to be solved.

Linear

Display if the problem is linear or nonlinear. Using 'on' the problem is treated as linear and 'off' for nonlinear.

numjac

Specify if the problem can be solved by using the analytic Jacobian (`numjac = 0`) or by computing it numerically (`numjac = 1`).

numbcjac

Specify if the problem can be solved by using the analytic partial derivative of the boundary condition (`numbcjac = 0`) or by computing it numerically (`numbcjac = 1`).

Vectorized

The solver can use the option `Vectorized` 'on' or 'off' to reduce the number of function evaluations required. By default, `Vectorized` is 'on'.

JVectorized

The solver can use the option `JVectorized` 'on' or 'off' to reduce the number of function evaluations required. By default, `JVectorized` is 'on'.

solinit

A guess for the initial mesh and the solution.

2.2 The function fun

This subroutine evaluates the function f .

```
function F = fun(X,Z,ExtraArgs)
```

Meaning of the arguments:

X

The time point where the function is evaluated.

Z

The value of y at which the function is evaluated.

ExtraArgs

Argument that declares the optional argument in the main program.

F

The resulting function value.

2.3 The function `bcfun`

This function evaluates the boundary condition.

```
function bc = bcfun(ya,yb,ExtraArgs)
```

Meaning of the arguments:

`ya`

Contains the left boundary condition $y(t_0)$.

`yb`

Contains the right boundary condition $y(t_n)$.

`ExtraArgs`

The same argument defined in the previous function.

`bc`

Contains the boundary value.

2.4 The function `dfun`

This function evaluates the analytic Jacobian of The ODE function.

```
function Df = dfun(X,Z,ExtraArgs)
```

Meaning of the arguments:

`X,Z`

The same arguments used in the function `fun`.

`Df`

The Jacobian value.

2.5 The function `dbcfun`

This function evaluates the partial derivatives of the boundary condition.

```
function [C0,C1] = dbcfun(ya,yb,ExtraArgs)
```

Meaning of the arguments:

`ya,yb`

The same arguments used in the function `bcfun`.

`ExtraArgs`

The same argument defined in the previous function.

`C0,C1`

Contains the partial derivatives of the boundary condition.

2.6 The function `settolerances`

This function defines the input tolerances `rtol`.

```
function tolvect = settolerances(tol)
```

Meaning of the arguments:

`tol`

Relative tolerance.

`tolvect`

A vector of tolerances that will be used in the code.

2.7 The function `setoutput`

This function contains information about the required output.

```
function [solref,printsolout,nindsol,indsol] = setoutput(plotsol)
```

Meaning of the arguments:

`plotsol`

Index of component to be plotted (optional).

`solref`

Contains information about the reference solution.

- `solref = 1`, means that the reference solution is available.

- `solref = 0`, means that the reference solution is not available.

`printsolout`

Contains information about the required output.

`nindsol`

If `printsolout = 1`, `nindsol` contains the number of components to be printed.

`indsol`

If `printsolout = 1`, `indsol(1:nindsol)` contains the index of the `nindsol` components to be printed.

2.8 The function `esolu`

This function evaluates the exact solution of the problem if it exists.

```
function Exact = esolu(X,ExtraArgs)
```

Meaning of the arguments:

`Exact`

Contains the exact solution.

`X`, `ExtraArgs`

The same argument as in function `fun`.

3 How to solve test problems with driver function

From the Matlab command line the driver function can be run by calling

```
>> [sol,time] = bvpMtest(problem,solver,tol,plotsol,opts,varargin);
```

The directory “problems” should be added to the MATLAB path before calling `bvpMtest`.

The input arguments are:

`problem`

The function handle of the problem.

`solver`

String with the calling name of the solver and there are six types in which the user can use.
`solver='twpbvp_m'`: Deferred correction scheme based on MIRK methods,
`solver='twpbvpc_m'`: Deferred correction scheme based on MIRK methods and conditioning,
`solver='twpbvp_1'`: Deferred correction scheme based on LOBATTO methods,
`solver='twpbvpc_1'`: Deferred correction scheme based on LOBATTO methods and conditioning,
`solver='acdc'`: Continuation strategy based on LOBATTO methods.
`solver='acdcc'`: Continuation strategy based on LOBATTO methods and conditioning.
`solver='bvp4c'`: Matlab solver `bvp4c`.
`solver='bvp5c'`: Matlab solver `bvp5c`.

`tol`

Relative tolerance.

`plotsol`

Index of component to be plotted (optional).

`opts`

Solver parameters set with `twpbvpcset` (optional).

`varargin`

Argument that declares the optional argument in the main program.

The output values are:

`sol`

The structure `sol` given by the solver with
`sol.x`: mesh point of the approximate solution
`sol.y`: solution computed at `sol.x`
`sol.solver`: solver used
`sol.error`: evaluate the error
`sol.mescd`: mixed significant digit of the approximate solution
`sol.iflbvp`: error flag (0 for successful run and 1 for an error)
`sol.condpar`: information about the conditioning parameters
`sol.stats`: Computational cost statistics and information about the maximum scaled error

`time`

Evaluate the execution time in solving the problem.

4 Linear Problems

4.1 Problem `bvpT1`

The problem is

$$\lambda z'' = z, \quad z(0) = 1, \quad z(1) = 0,$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$, and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1) \end{pmatrix},$$

where

$$f(z) = z,$$

and

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Exact solution

$$z(t) = (\exp(-t/\sqrt{\lambda}) - \exp((t-2)/\sqrt{\lambda})) / (1 - \exp(-2/\sqrt{\lambda})).$$

This problem is `bvpT1` in the `twpbvpc` package and we can run it by calling the driver function in command line as follows:

```
>> opts= twpbvpcset('nmax',10000);  
>> [sol,time]= bvpMtest(@bvpT1,'twpbvpc_1',1e-2,1,opts,1e-3);
```

solves the problem 1 (`bvpT1`) in the `twpbvpc` package using the solver '`twpbvpc_1`', maximum number of mesh points = 10000, relative tolerance equal to 1e-2, and the the parameter $\lambda = 1e-3$. The output variable `sol` contains information about the solution and `time` is the execution time for solving the problem.

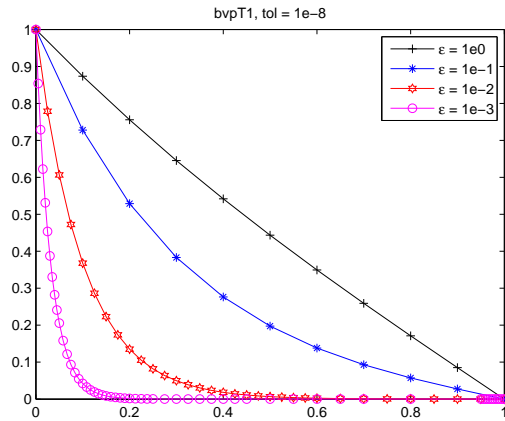


Figure 1: Behavior of the solution of problem 4.1.

4.2 Problem bvpT2

The problem is

$$\lambda z'' = z', \quad z(0) = 1, \quad z(1) = 0,$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$, and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_2) \end{pmatrix},$$

where

$$f(z') = z',$$

and

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Exact solution

$$z(t) = (1 - \exp((t-1)/\lambda)) / (1 - \exp(-1/\lambda)).$$

This problem bvpT2 can be implemented as follows:

```
>> opts= twpbvpcset('nmax',10000);
>> [sol,time]= bvpMtest(@bvpT2,'twpbvpc_1',1e-4,1,opts,1e-3);
```

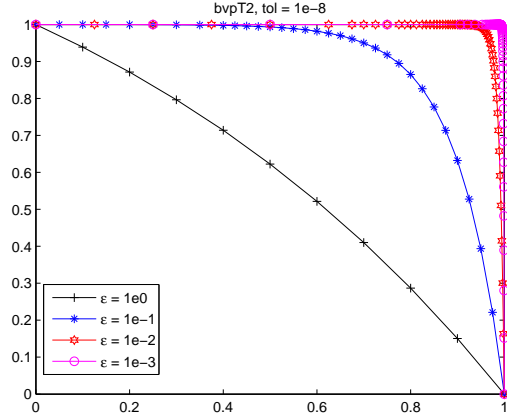



Figure 2: Behavior of the solution of problem 4.2.

4.3 Problem bvpT3

The problem is

$$\lambda z'' = -(2 + \cos(\pi t))z' + z - (1 + \lambda\pi^2)\cos(\pi t) - (2 + \cos(\pi t))\pi \sin(\pi t), \quad z(-1) = -1 \quad z(1) = -1,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda}f(t, y_1, y_2) \end{pmatrix},$$

where

$$f(t, z, z') = -(2 + \cos(\pi t))z' + z - (1 + \lambda\pi^2)\cos(\pi t) - (2 + \cos(\pi t))\pi \sin(\pi t),$$

and

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t).$$

The problem has no turning points and the solution is smooth.

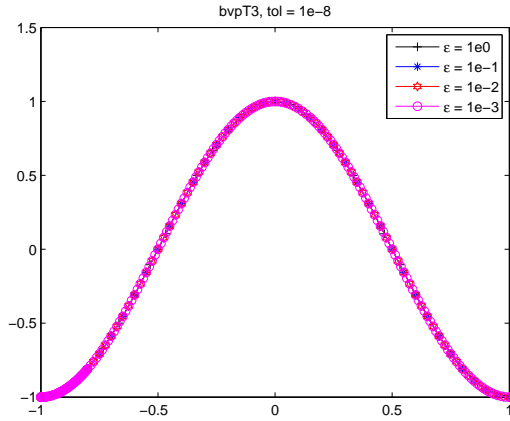


Figure 3: Behavior of the solution of problem 4.3.

4.4 Problem bvpT4

The problem is

$$\lambda z'' = -z' + (1 + \lambda)z, \quad z(-1) = 1 + \exp(-2), \quad z(1) = 1 + \exp(-2(1 + \lambda)/\lambda),$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -z' + (1 + \lambda)z,$$

and

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 + \exp(-2) \\ 1 + \exp(-2(1 + \lambda)/\lambda) \end{pmatrix}.$$

Exact solution

$$z(t) = \exp(t - 1) + \exp(-(1 + \lambda)(1 + t)/\lambda).$$

The solution has a boundary layer of width $O(\epsilon)$ at $t = -1$.

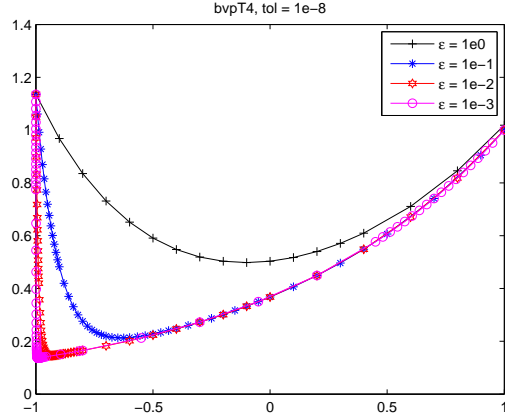


Figure 4: Behavior of the solution of problem 4.4.

4.5 Problem bvpT5

The problem is

$$\lambda z'' = tz' + z - (1 + \lambda\pi^2) \cos(\pi t) + \pi t \sin(\pi t), \quad z(-1) = -1, \quad z(1) = -1,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1, y_2) \end{pmatrix},$$

where

$$f(t, z, z') = tz' + z - (1 + \lambda\pi^2) \cos(\pi t) + \pi t \sin(\pi t),$$

and

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t).$$

The problem has a turning point at $t = 0$ but the solution is smooth.

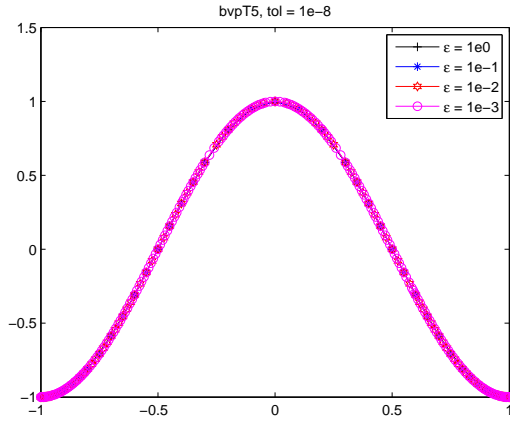


Figure 5: Behavior of the solution of problem 4.5.

4.6 Problem bvpT6

The problem is

$$\lambda z'' = -tz' - \lambda\pi^2 \cos(\pi t) - \pi t \sin(\pi t), \quad z(-1) = -2, \quad z(1) = 0,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_2) \end{pmatrix},$$

where

$$f(t, z') = -tz' - \lambda\pi^2 \cos(\pi t) - \pi t \sin(\pi t),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t) + \operatorname{erf}(t/\sqrt{2\lambda})/\operatorname{erf}(1/\sqrt{2\lambda}).$$

The solution has a shock layer in the turning point region near $t = 0$.

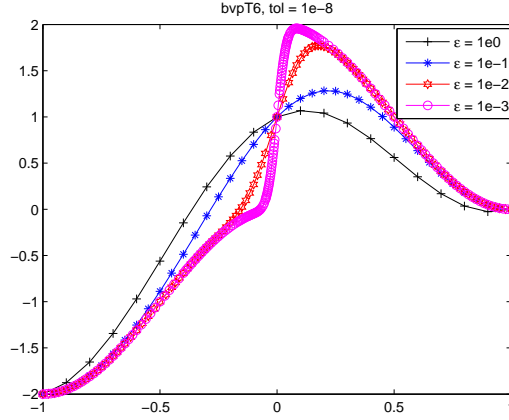


Figure 6: Behavior of the solution of problem 4.6.

4.7 Problem bvpT7

The problem is

$$\lambda z'' = -tz' + z - (1 + \lambda\pi^2) \cos(\pi t) - \pi t \sin(\pi t), \quad z(-1) = -1, \quad z(1) = 1,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1, y_2) \end{pmatrix},$$

where

$$f(t, z, z') = -tz' + z - (1 + \lambda\pi^2) \cos(\pi t) - \pi t \sin(\pi t),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t) + t + \frac{t \operatorname{erf}(t/\sqrt{2\lambda}) + \sqrt{2\lambda/\pi} \exp(-t^2/2\lambda)}{\operatorname{erf}(1/\sqrt{2\lambda}) + \sqrt{2\lambda/\pi} \exp(-1/2\lambda)}.$$

The solution has a corner layer in the turning point region near $t = 0$.

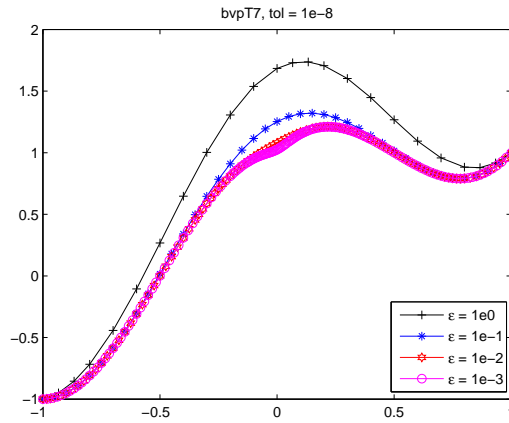


Figure 7: Behavior of the solution of problem 4.7.

4.8 Problem bvpT8

The problem is

$$\lambda z'' = -z', \quad z(0) = 1, \quad z(1) = 2,$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_2) \end{pmatrix},$$

where

$$f(z') = -z',$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Exact solution

$$z(t) = (2 - \exp(-1/\lambda) - \exp(-t/\lambda))/(1 - \exp(-1/\lambda)).$$

The solution has a boundary layer of width $O(\lambda)$ at $t = 0$.

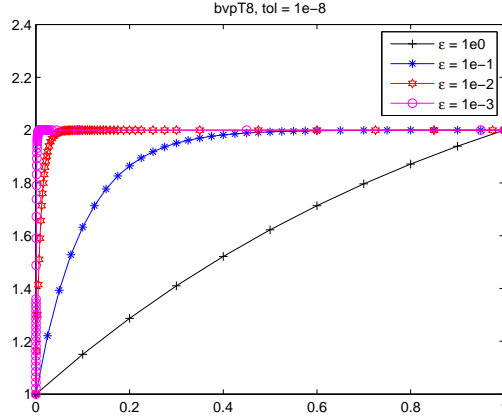


Figure 8: Behavior of the solution of problem 4.8.

4.9 Problem bvpT9

The problem is

$$(\lambda + t^2)z'' = -4tz' - 2z, \quad z(-1) = 1/(1 + \lambda), \quad z(1) = 1/(1 + \lambda),$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda + t^2} f(t, y_1, y_2) \end{pmatrix},$$

where

$$f(t, z, z') = -4tz' - 2z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1/(1 + \lambda) \\ 1/(1 + \lambda) \end{pmatrix}.$$

Exact solution

$$z(t) = 1/(\lambda + x^2).$$

The solution has a spike of height $\frac{1}{\lambda}$ at $t = 0$.

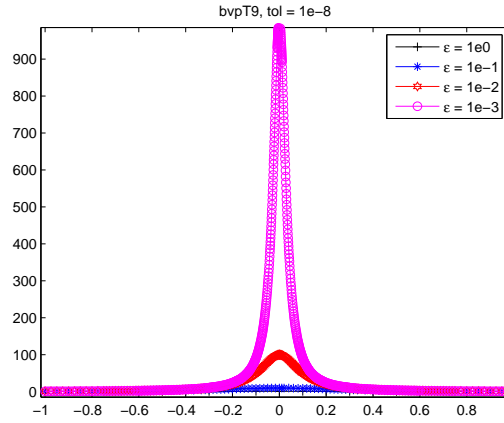


Figure 9: Behavior of the solution of problem 4.9.

4.10 Problem `bvpT10`

The problem is

$$\lambda z'' = -tz', \quad z(-1) = 0, \quad z(1) = 2,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_2) \end{pmatrix},$$

where

$$f(t, z') = -tz',$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Exact solution

$$z(t) = 1 + \operatorname{erf}(t/\sqrt{2\lambda})/\operatorname{erf}(1/\sqrt{2\lambda}).$$

The solution has a turning point of width $O(\sqrt{\lambda})$ at $t = 0$.

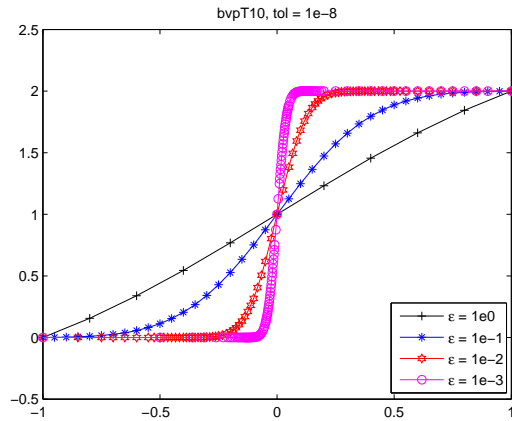


Figure 10: Behavior of the solution of problem 4.10.

4.11 Problem bvpT11

The problem is

$$\lambda z'' = z - (1 + \lambda\pi^2) \cos(\pi t), \quad z(-1) = -1, \quad z(1) = -1,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = z - (1 + \lambda\pi^2) \cos(\pi t),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t).$$

The solution is smooth.

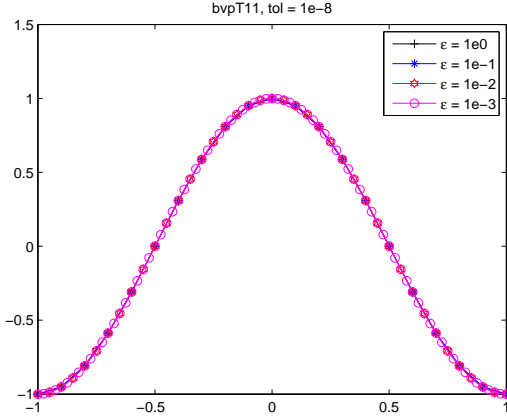


Figure 11: Behavior of the solution of problem 4.11.

4.12 Problem bvpT12

The problem is

$$\lambda z'' = z - (1 + \lambda\pi^2) \cos(\pi t), \quad z(-1) = -1, \quad z(1) = 0,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = z - (1 + \lambda\pi^2) \cos(\pi t),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t) + \frac{\exp((t+1)/\sqrt{\lambda}) - \exp((-t-1)/\sqrt{\lambda})}{\exp(2/\sqrt{\lambda}) - \exp(-2/\sqrt{\lambda})}.$$

The solution has a turning point of width $O(\sqrt{\lambda})$ at $t = 1$.

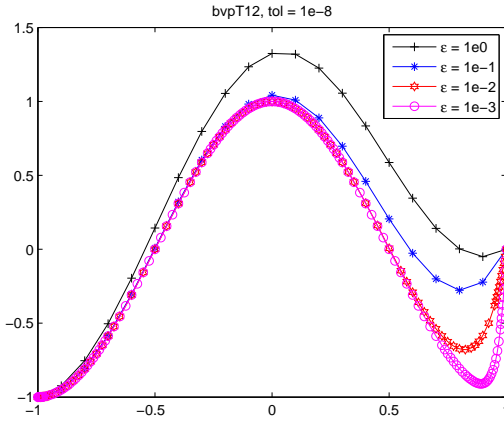


Figure 12: Behavior of the solution of problem 4.12.

4.13 Problem bvpT13

The problem is

$$\lambda z'' = z - (1 + \lambda\pi^2) \cos(\pi t), \quad z(-1) = 0, \quad z(1) = 1,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = z - (1 + \lambda\pi^2) \cos(\pi t).$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t) + \exp(-(t+1)/\sqrt{\lambda}).$$

The solution has a boundary layer of width $O(\sqrt{\lambda})$ near $t = -1$.

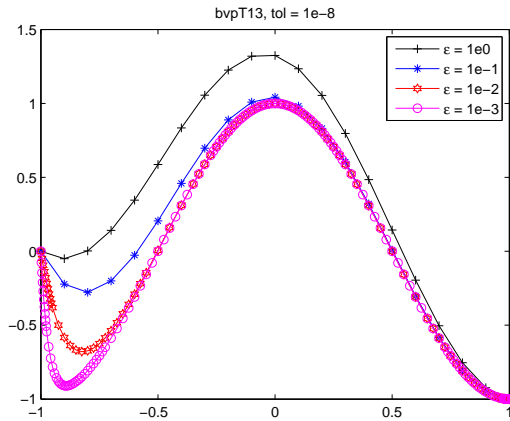


Figure 13: Behavior of the solution of problem 4.13.

4.14 Problem bvpT14

The problem is

$$\lambda z'' = z - (1 + \lambda\pi^2) \cos(\pi t), \quad z(-1) = 0, \quad z(1) = 0,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = z - (1 + \lambda\pi^2) \cos(\pi t),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exact solution

$$z(t) = \cos(\pi t) + \exp((t-1)/\sqrt{\lambda}) + \exp(-(t+1)/\sqrt{\lambda}).$$

The solution has a boundary layer of width $O(\sqrt{\lambda})$ near $t = -1$ and $t = 1$.

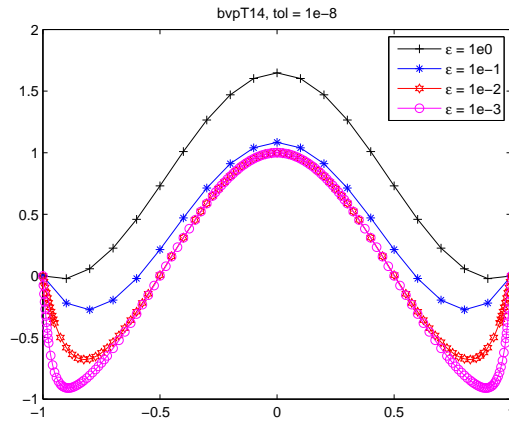


Figure 14: Behavior of the solution of problem 4.14.

4.15 Problem bvpT15

The problem is

$$\lambda z'' = tz, \quad z(-1) = 1, \quad z(1) = 1,$$

with

$$z \in \mathbb{R}, \quad t \in [-1, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = tz,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-1, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

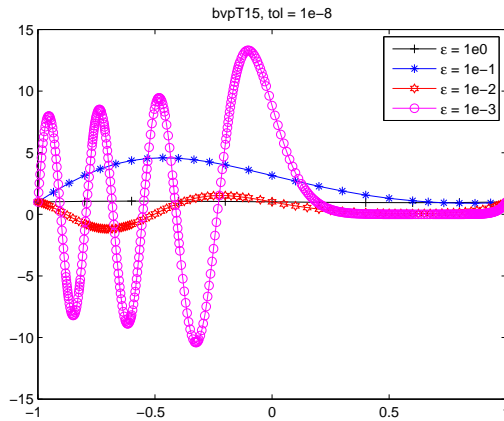


Figure 15: Behavior of the solution of problem 4.15.

4.16 Problem bvpT16

The problem is

$$\lambda^2 z'' = -\pi^2 z/4, \quad z(0) = \sin(\pi/2\lambda), \quad z(1) = \sin(\pi/2\lambda),$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda^2} f(y_1) \end{pmatrix},$$

where

$$f(z) = -\pi^2 z/4,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} \sin(\pi/2\lambda) \\ \sin(\pi/2\lambda) \end{pmatrix}.$$

Exact solution

$$z(t) = \sin(\pi t/2\lambda) \text{ when } 1/\lambda \text{ is odd.}$$

The solution oscillates rapidly for small λ .

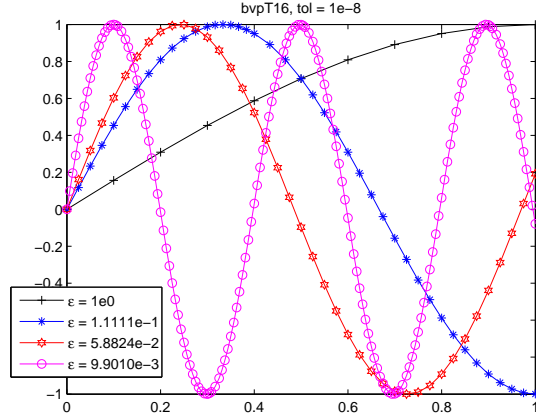


Figure 16: Behavior of the solution of problem 4.16.

4.17 Problem bvpT17

The problem is

$$z'' = -3\lambda z/(\epsilon + t^2)^2, \quad z(-0.1) = -0.1/\sqrt{\lambda + 0.01}, \quad z(0.1) = 0.1/\sqrt{\lambda + 0.01}$$

with

$$z \in \mathbb{R}, \quad t \in [-0.1, 0.1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = -3\lambda z/(\lambda + t^2)^2,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [-0.1, 0.1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -0.1/\sqrt{\lambda + 0.01} \\ 0.1/\sqrt{\lambda + 0.01} \end{pmatrix}.$$

Exact solution

$$z(t) = t/\sqrt{\lambda + t^2}.$$

The solution has a boundary layer of width $O(\sqrt{\lambda})$ at $t = 0$.

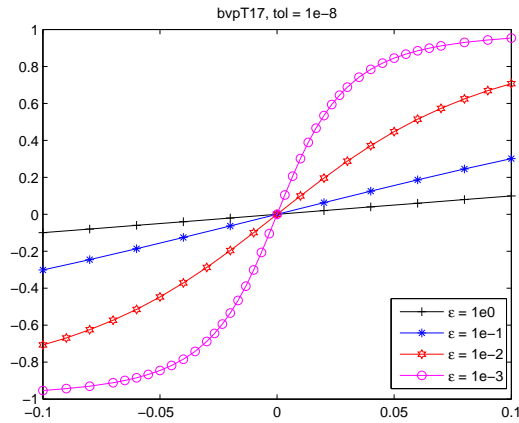


Figure 17: Behavior of the solution of problem 4.17.

4.18 Problem bvpT18

The problem is

$$\lambda z'' = -z', \quad z(0) = 1, \quad z(1) = \exp(-1/\lambda)$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_2) \end{pmatrix},$$

where

$$f(z') = -z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ \exp(-1/\lambda) \end{pmatrix}.$$

Exact solution

$$z(t) = \exp(-t/\lambda).$$

The solution has a boundary layer of width $O(\lambda)$ at $t = 0$.

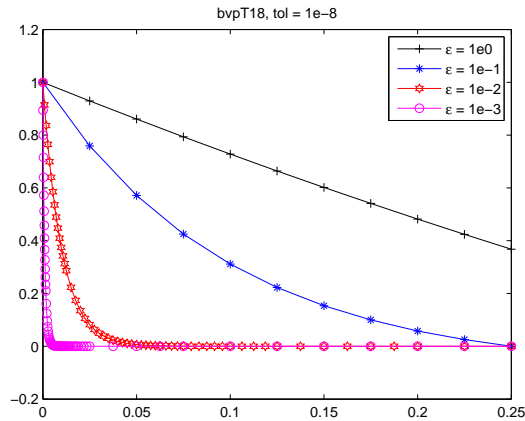


Figure 18: Behavior of the solution of problem 4.18.

5 Nonlinear Problems

5.1 Problem bvpT19

The problem is

$$\lambda z'' = -\exp(z)z' + \frac{\pi}{2} \sin(\pi t/2) \exp(2z), \quad z(0) = 0, \quad z(1) = 0$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1, y_2) \end{pmatrix},$$

where

$$f(t, z, z') = -\exp(z)z' + \frac{\pi}{2} \sin(\pi t/2) \exp(2z),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

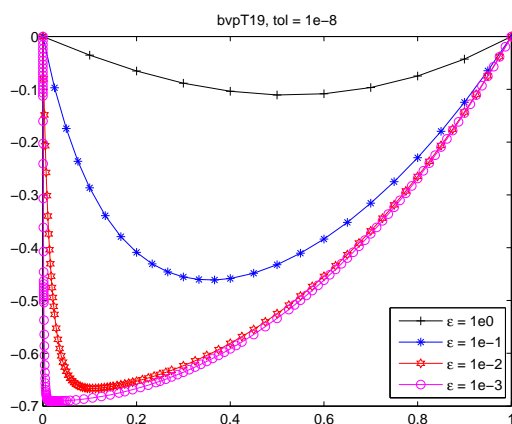


Figure 19: Behavior of the solution of problem 5.1.

5.2 Problem bvpT20

The problem is

$$\lambda z'' = -(z')^2 + 1, \quad z(0) = 1 + \lambda \ln \cosh(-0.745/\lambda), \quad z(1) = 1 + \lambda \ln \cosh(0.255/\lambda)$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_2) \end{pmatrix},$$

where

$$f(z') = -(z')^2 + 1,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 + \lambda \ln \cosh(-0.745/\lambda) \\ 1 + \lambda \ln \cosh(0.255/\lambda) \end{pmatrix}.$$

Exact solution

$$z(t) = 1 + \lambda \ln \cosh(t - 0.745/\lambda).$$

The solution has a corner layer at $t = 0.745$.

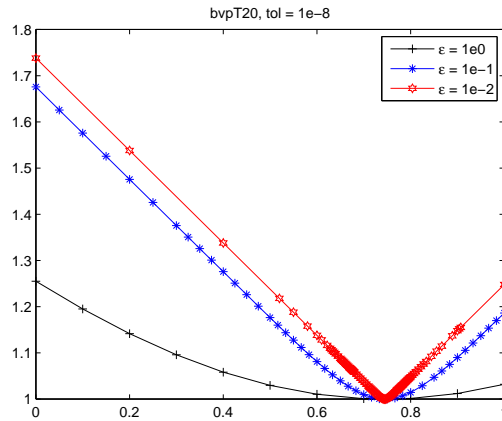


Figure 20: Behavior of the solution of problem 5.2.

5.3 Problem bvpT21

The problem is

$$\lambda z'' = z + z^2 - \exp(-2t/\sqrt{\lambda}), \quad z(0) = 1, \quad z(1) = \exp(-1/\sqrt{\lambda})$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(t, y_1) \end{pmatrix},$$

where

$$f(t, z) = z + z^2 - \exp(-2t/\sqrt{\lambda}),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ \exp(-1/\sqrt{\lambda}) \end{pmatrix}.$$

Exact solution

$$z(t) = \exp(-t/\sqrt{\lambda}).$$

The solution has a boundary layer of width $O(\sqrt{\lambda})$ at $t = 0$.

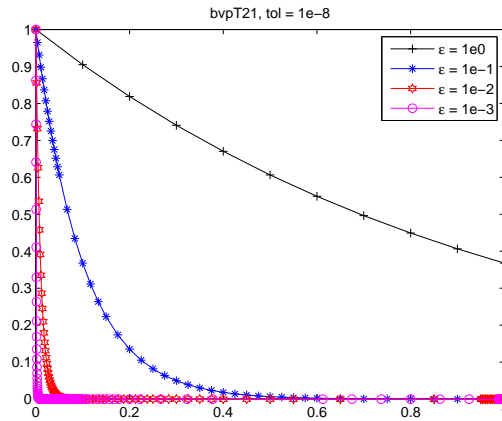


Figure 21: Behavior of the solution of problem 5.3.

5.4 Problem bvpT22

The problem is

$$\lambda z'' = -z' - z^2, \quad z(0) = 0, \quad z(1) = \frac{1}{2}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -z' - z^2,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

The solution has a boundary layer near $t = 0$.

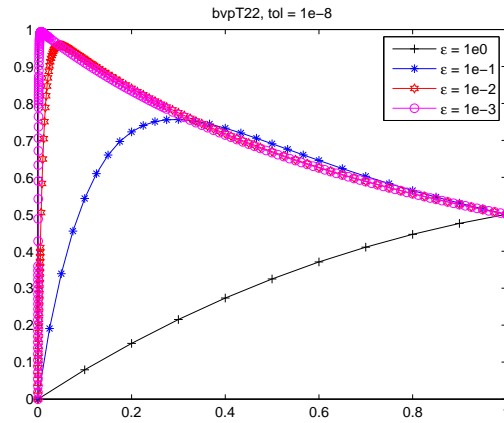


Figure 22: Behavior of the solution of problem 5.4.

5.5 Problem bvpT23

The problem is

$$z'' = \lambda \sinh(\lambda z), \quad z(0) = 0, \quad z(1) = 1$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ f(y_1) \end{pmatrix},$$

where

$$f(z) = \lambda \sinh(\lambda z),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution has a boundary layer near $t = 1$.

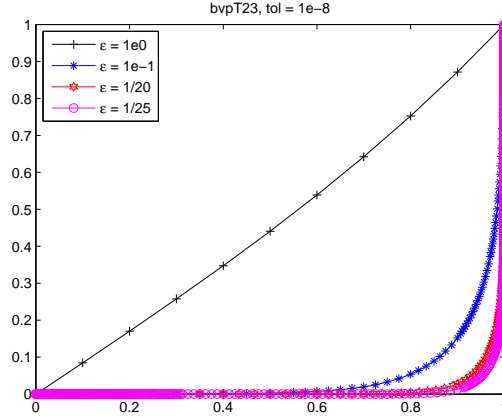


Figure 23: Behavior of the solution of problem 5.5.

5.6 Problem bvpT24

The problem is

$$\lambda A(t)zz'' = \left(\frac{1+\gamma}{2} - \lambda A'(t)\right)zz' - \frac{z'}{z} - \frac{A'(t)}{A(t)}\left(1 - \left(\frac{\gamma-1}{2}\right)z^2\right),$$

$$A(t) = 1 + t^2, \quad \gamma = 1.4, \quad z(0) = 0.9129, \quad z(1) = 0.375$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda A(t)y_1} f(t, y_1, y_2) \end{pmatrix},$$

where

$$f(t, z, z') = \left(\frac{1+\gamma}{2} - \lambda A'(t)\right)zz' - \frac{z'}{z} - \frac{A'(t)}{A(t)}\left(1 - \left(\frac{\gamma-1}{2}\right)z^2\right),$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0.9129 \\ 0.375 \end{pmatrix}.$$

This equation represents a shock wave in a one dimensional nozzle flow.

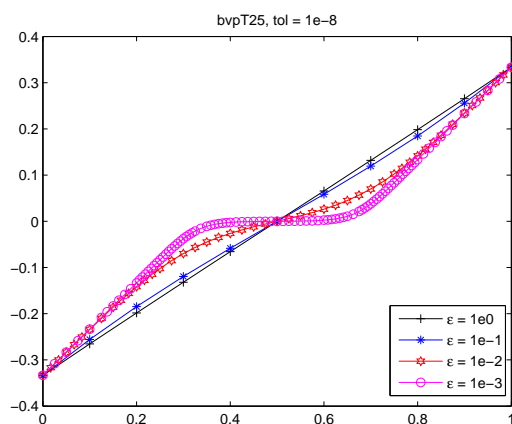


Figure 24: Behavior of the solution of problem 5.6.

5.7 Problem bvpT25

The problem is

$$\lambda z'' = -zz' + z, \quad z(0) = -\frac{1}{3}, \quad z(1) = \frac{1}{3}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -zz' + z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

The solution has corner layers at $t = \frac{1}{3}$ and $t = \frac{2}{3}$.

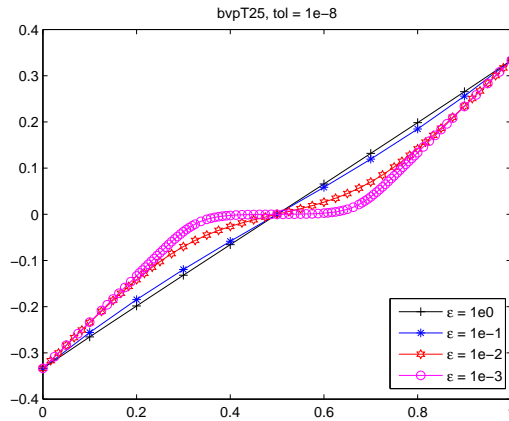


Figure 25: Behavior of the solution of problem 5.7.

5.8 Problem bvpT26

The problem is

$$\lambda z'' = -zz' + z, \quad z(0) = 1, \quad z(1) = -\frac{1}{3}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -zz' + z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}.$$

The solution has corner layers at $t = 0$ and $t = 1$.

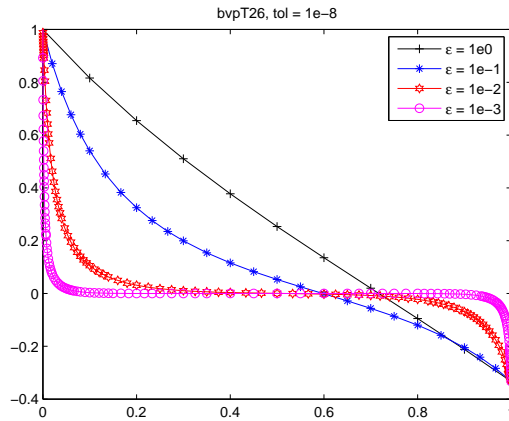


Figure 26: Behavior of the solution of problem 5.8.

5.9 Problem bvpT27

The problem is

$$\lambda z'' = -zz' + z, \quad z(0) = 1, \quad z(1) = \frac{1}{3}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -zz' + z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}.$$

The solution has corner layers at $t = 0$ and a corner layer at $t = \frac{2}{3}$.

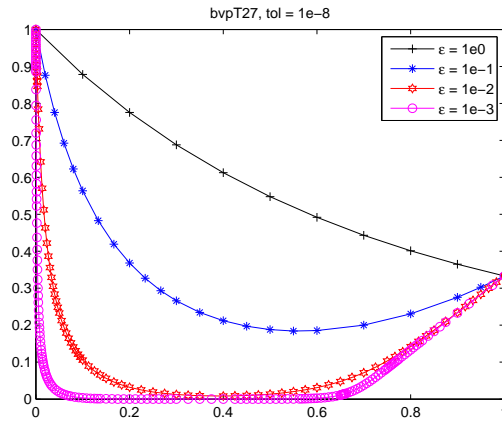


Figure 27: Behavior of the solution of problem 5.9.

5.10 Problem bvpT28

The problem is

$$\lambda z'' = -zz' + z, \quad z(0) = 1, \quad z(1) = \frac{3}{2}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -zz' + z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}.$$

The solution has corner layers at $t = 0$.

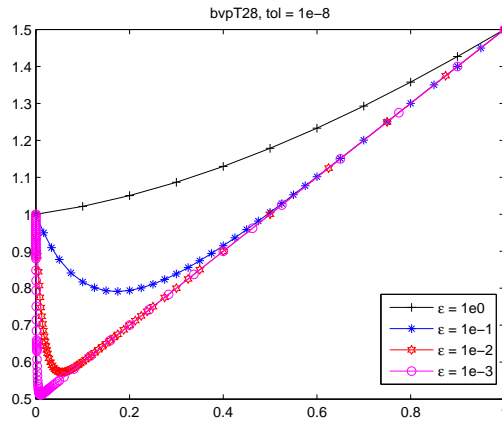


Figure 28: Behavior of the solution of problem 5.10.

5.11 Problem bvpT29

The problem is

$$\lambda z'' = -zz' + z, \quad z(0) = 0, \quad z(1) = \frac{3}{2}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -zz' + z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}.$$

The solution has corner layers at $t = 0$.

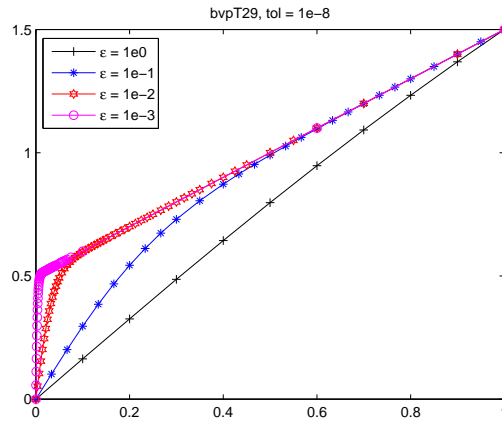


Figure 29: Behavior of the solution of problem 5.11.

5.12 Problem bvpT30

The problem is

$$\lambda z'' = -zz' + z, \quad z(0) = -\frac{7}{6}, \quad z(1) = \frac{3}{2}$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$ and $y_2 = z'$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} f(y_1, y_2) \end{pmatrix},$$

where

$$f(z, z') = -zz' + z,$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} = \begin{pmatrix} -\frac{7}{6} \\ \frac{3}{2} \end{pmatrix}.$$

The solution has a shock layer at $t = \frac{1}{3}$.

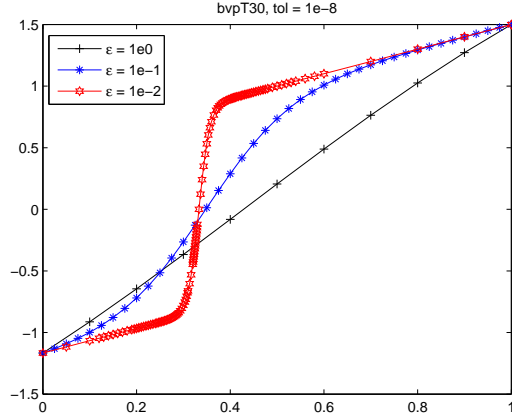


Figure 30: Behavior of the solution of problem 5.12.

5.13 Problem bvpT31

The problem is

$$z' = \sin \theta, \theta' = M, \lambda M' = -Q, \lambda Q' = (z - 1) \cos \theta - MT, T = \sec \theta + \lambda Q \tan \theta, \\ z(0) = z(1) = 0, M(0) = M(1) = 0.$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$, $y_2 = \theta$, $y_3 = M$ and $y_4 = Q$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' = \begin{pmatrix} \sin y_2 \\ y_3 \\ -y_4/\lambda \\ f(y_1, y_2, y_3, y_4) \end{pmatrix},$$

where

$$f(z, \theta, M, Q) = \frac{1}{\lambda}((z - 1) \cos \theta - M \sec \theta) + \lambda Q \tan \theta$$

with

$$(y_1, y_2, y_3, y_4)^T \in \mathbb{R}^4, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \\ y_4(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This equation models nonlinear elastic beams.

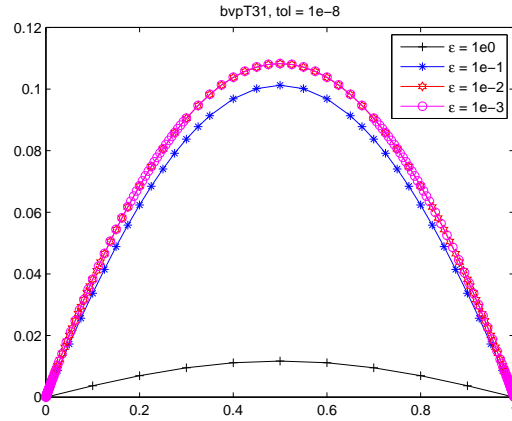


Figure 31: Behavior of the solution of problem 5.13.

5.14 Problem bvpT32

The problem is

$$z'''' = \lambda(z'z'' - zz'''), \quad z(0) = z'(0) = 0, \quad z(1) = 1, \quad z'(1) = 0$$

with

$$z \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = z$, $y_2 = z'$, $y_3 = z''$ and $y_4 = z'''$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ f(y_1, y_2, y_3, y_4) \end{pmatrix},$$

where

$$f(z, z', z'', z''') = \lambda(z'z'' - zz'''),$$

with

$$(y_1, y_2, y_3, y_4)^T \in \mathbb{R}^4, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \\ y_4(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

This problem arises from fluid injection through one side of a long vertical channel

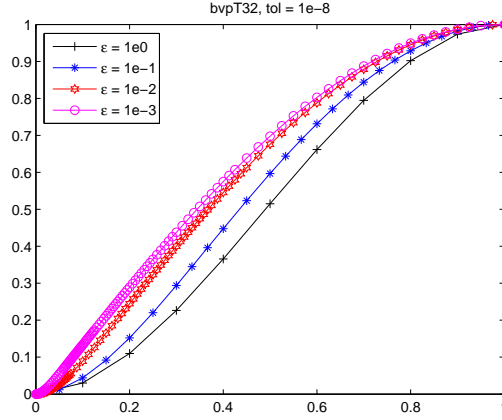


Figure 32: Behavior of the solution of problem 5.14.

5.15 Problem bvpT33

The problem is

$$\lambda z'''' = -zz''' - yy', \quad \lambda y'' = yz' - zy', \quad y(0) = -1, y(1) = 1, \quad z(0) = z'(0) = z(1) = z'(1) = 0,$$

with

$$z, y \in \mathbb{R}, \quad t \in [0, 1].$$

We write this problem in first order form by defining $y_1 = y$, $y_2 = y'$, $y_3 = z$, $y_4 = z'$, $y_5 = z''$ and $y_6 = z'''$, yielding a system of differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \frac{1}{\lambda} y_1 y_4 - y_3 y_2 \\ y_4 \\ y_5 \\ y_6 \\ \frac{1}{\lambda} (-y_3 y_6 - y_1 y_2) \end{pmatrix},$$

with

$$(y_1, y_2, y_3, y_4, y_5, y_6)^T \in \mathbb{R}^6, \quad t \in [0, 1].$$

The boundary conditions are obtained from

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(1) \\ y_2(1) \\ y_3(1) \\ y_4(1) \\ y_5(1) \\ y_6(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

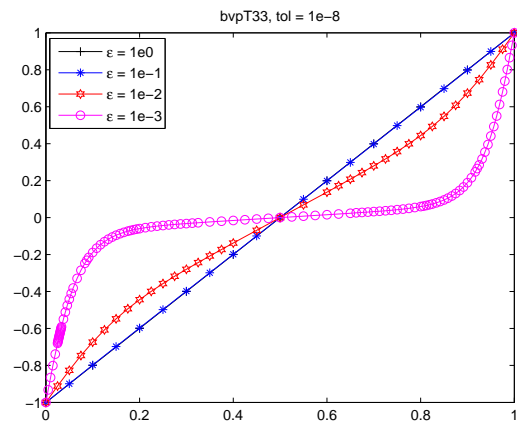


Figure 33: Behavior of the solution of problem 5.15.