CSC2302H

Assignment 3

March 4, 2014 University of Toronto **Due:** April 1, 2014

From 'Solving Ordinary Differential Equations II', E. Hairer and G. Wanner, Springer 1991, p. 163.

You are to investigate, by numerical experiments using a method suitable for stiff systems, the "circular nerve model" defined by the system of ODEs developed below. You are to use a stiff solver such as ode15s (from matlab) or the Fortran code RADAU (available from a link on the course webpage). In particular you are to show that this system loses its limit cycle when the diffusion coefficient D becomes either too large or too small. This system of ODEs models a combination of a threshold-nerve-impulse mechanism, a cusp catastrophe

$$\epsilon y^{'} = -(y^3 + ay + b),$$

(with a "smooth return" – see Zeeman, 1972 reference of HW), and a Van der Pol oscillator to keep the solution away from the origin. The unknown functions y, a and b are each functions of time and space, where $y(t, x)$ is the value of the nerve impulse at time $t \geq 0$ at location x associated with a one-dimensional nerve ($0 \leq x \leq 1$).

$$\begin{aligned}
\frac{\partial y}{\partial t} &= -\frac{1}{\epsilon}(y^3 + ay + b) + \sigma \frac{\partial^2 y}{\partial x^2} \\
\frac{\partial a}{\partial t} &= b + 0.07v + \sigma \frac{\partial^2 a}{\partial x^2} \\
\frac{\partial b}{\partial t} &= (1 - a^2)b - a - 0.4y + .035v + \sigma \frac{\partial^2 b}{\partial x^2}
\end{aligned}$$

where

$$v = \frac{u}{u + 0.1}\ , \ u = (y - 0.7)(y - 1.3).$$

We consider discretizing the space dimension using a uniform mesh, ($0 = x_0 < x_1 \cdots x_N = 1$), with $x_i = i * \Delta x$. When the partial derivatives with respect to x are replaced by finite differences (for example, $\frac{\partial^2 y}{\partial x^2}|_{x_i}$ is replaced by $\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$) this model becomes a system of ODEs in time (with $y_i(t)$ being an approximation to the one dimensional function $y(t, x_i)$). We let the "nerve" be closed like a torus so that the nerve impulse goes around without stopping. (That is, for any $t \geq 0$ we assume $y(t, 1) = y(t, 0)$, $a(t, 1) = a(t, 0)$

and $b(t,1) = b(t,0)$.) The Jacobian of the resulting system is then sparse, although not banded. Stiffness in this problem has two sources: firstly the parameter ϵ becoming small, secondly the diffusion term D becoming large for small discretization intervals Δx.

For example with $\epsilon = 10^{-4}$, $\sigma = 1/144$, $0 \le x \le 1$, $\Delta x = 1/32$ and $N = 32$, we obtain

$$\begin{aligned} y_i^{'} &= -10^4(y_i^3 + a_i y_i + b_i) + D(y_{i-1} - 2y_i + y_{i+1}) \\ a_i^{'} &= b_i + 0.07v_i + D(a_{i-1} - 2a_i + a_{i+1}) \\ b_i^{'} &= (1 - a_i^2)b_i - a_i - 0.4y_i + 0.035v_i + D(b_{i-1} - 2b_i + b_{i+1}) \end{aligned}$$

for $i = 1, \cdots N$, where

$$v_i = \frac{u_i}{u_i + 0.1}, \;\; u_i = (y_i - 0.7)(y_i - 1.3), \;\; D = N^2\sigma = \frac{N^2}{144},$$

and the required "boundary conditions" (to define the finite differences near the endpoint values of x) are

$$y_0 = y_N, \;\; a_0 = a_N, \;\; b_0 = b_N,$$

$$y_{N+1} = y_1, \;\; a_{N+1} = a_1, \;\; b_{N+1} = b_1.$$

This defines a system of ODEs of dimension $3N = 96$. The initial values are

$$y_i(0) = 0, \; a_i(0) = -2cos(\frac{2i\pi}{N}), \; b_i(0) = 2sin(\frac{2i\pi}{N}), \quad \text{for } i = 1 \cdots N.$$

1. For this particular discretization and choice of parameters you are to solve this problem using a Stiff solver with and without supplying analytic derivatives for the Jacobian matrix. In your write-up discuss whether, on this problem, the extra effort required to supply the analytic Jacobian is reflected in reduced costs, improved accuracy, or improved robustness.
2. By experimenting with different values of σ (and possibly N) show that this system loses its limit cycle when D becomes too large or too small). (Note that the system of ODEs given above for $N = 32$, will change if N is increased.)