CSC2302H

Assignment 3

November 9, 2010 University of Toronto **Due:** December 7, 2010

From 'Solving Ordinary Differential Equations II', E. Hairer and G. Wanner, Springer 1991, p. 163.

You are to investigate, by numerical experiments using a method suitable for stiff systems, the "circular nerve model" defined by the system of ODEs developed below. You are to use a stiff solver such as ode15s (from matlab) or the Fortran code RADAU (available from a link on the course webpage). In particular you are to show that this system loses its limit cycle when the diffusion coefficient D becomes either too large or too small. This system of ODEs models a combination of a threshold-nerve-impulse mechanism, a cusp catastrophe

$$\epsilon y^{'} = -(y^3 + ay + b),$$

(with a "smooth return" – see Zeeman, 1972 reference of HW), and a Van der Pol oscillator to keep the solution away from the origin. The unknown functions y, a and b are each functions of time and space, where $y(t, x)$ is the value of the nerve impulse at time $t \geq 0$ at location x associated with a one-dimensional nerve ($0 \leq x \leq 1$).

$$\begin{aligned}
\frac{\partial y}{\partial t} &= -\frac{1}{\epsilon}(y^3 + ay + b) + \sigma \frac{\partial^2 y}{\partial x^2} \\
\frac{\partial a}{\partial t} &= b + 0.07v + \sigma \frac{\partial^2 a}{\partial x^2} \\
\frac{\partial b}{\partial t} &= (1 - a^2)b - a - 0.4y + .035v + \sigma \frac{\partial^2 b}{\partial x^2}
\end{aligned}$$

where

$$v = \frac{u}{u + 0.1}\ ,\ u = (y - 0.7)(y - 1.3).$$

We consider discretizing the space dimension using $0 = x_0 < x_1 \cdots x_N = 1$ with $x_i = i * \Delta x$. When the partial derivatives with respect to x are replaced by finite differences (for example, $\frac{\partial^2 y}{\partial x^2}|_{x_i}$ is replaced by $\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$) this model becomes a system of ODEs in time (with $y_i(t)$ being an approximation to the one dimensional function $y(t, x_i)$). We let the "nerve" be closed like a torus so that the nerve impulse goes around without stopping. (That is, for any $t \geq 0$ we assume $y(t, 1) = y(t, 0)$, $a(t, 1) = a(t, 0)$ and $b(t, 1) = b(t, 0)$.) The Jacobian of the resulting system is then sparse, although not banded. Stiffness in this problem has two sources: firstly the parameter ϵ becoming small, secondly the diffusion term for small discretization intervals Δx.

With $\epsilon = 10^{-4}$, $\sigma = 1/144$, $0 \le x \le 1$, $\Delta x = 1/32$ and $N = 32$, we obtain

$$\begin{aligned} y_i^{'} &= -10^4(y_i^3 + a_i y_i + b_i) + D(y_{i-1} - 2y_i + y_{i+1}) \\ a_i^{'} &= b_i + 0.07v_i + D(a_{i-1} - 2a_i + a_{i+1}) \\ b_i^{'} &= (1 - a_i^2)b_i - a_i - 0.4y_i + 0.035v_i + D(b_{i-1} - 2b_i + b_{i+1}) \end{aligned}$$

for $i = 1, \cdots N$, where

$$v_i = \frac{u_i}{u_i + 0.1}, \;\; u_i = (y_i - 0.7)(y_i - 1.3), \;\; D = \frac{N^2}{144} = N^2\sigma,$$

and the required "boundary conditions" (to define the finite differences near the endpoint values of x) are

$$y_0 = y_N, \;\; a_0 = a_N, \;\; b_0 = b_N,$$

$$y_{N+1} = y_1, \;\; a_{N+1} = a_1, \;\; b_{N+1} = b_1.$$

This defines a system of ODEs of dimension $3N = 96$. The initial values are

$$y_i(0) = 0, \;\; a_i(0) = -2cos(\frac{2i\pi}{N}), \;\; b_i(0) = 2sin(\frac{2i\pi}{N}), \quad \text{for } i = 1 \cdots N.$$

You are to solve this problem using a Stiff solver with and without supplying analytic derivatives for the Jacobian matrix. In your write-up discuss whether, on this problem, the extra effort required to supply the analytic Jacobian is reflected in reduced costs, improved accuracy, or improved robustness.