CSC2302H

Assignment 2

February 4, 2014 University of Toronto **Due:** March 4, 2014

1. You are to investigate how well the initial value methods ode45 and ode113 of Matlab preserve some important properties of a system of IVPs that arises in classical mechanics. Consider the motion of two bodies under mutual gravitational attraction, where a body of mass m is orbiting a second body of much larger mass M. For example, the earth orbiting the sun or the moon orbiting the earth.

   From Newtons laws of motion and gravitation, the orbital trajectory $(x(t), y(t))$ can be described by a system of second order, ordinary differential equations,

   $$x'' = -GMx/r^3, \;\; y'' = -GMy/r^3,$$

   where G is the gravitational constant and $r = (x^2 + y^2)^{1/2}$ is the distance of the orbiting body from the center of mass of the two bodies. For this problem we have chosen units so that $GM = 1$.

   (a) Use ode45 and ode113 to solve this system of ODEs with the initial conditions,

   $$x(0) = 1 - \epsilon, \; y(0) = 0, \; x'(0) = 0, \; y'(0) = (\frac{1+\epsilon}{1-\epsilon})^{1/2},$$

   where ϵ is the eccentricity of the resulting elliptical orbit, which has period 2π. Try the values $\epsilon = 0$ (which should give a circular orbit), $\epsilon = 0.5$, and $\epsilon = 0.9$. For each case and for each of ode45 and ode113, solve the IVP for at least one period and produce a smooth plot of the orbital trajectory as well as a plot of x versus t and y versus t. Experiment with different values of the error tolerance to see how this affects the cost of the integration and how it affects the ability of the method to produce an approximate solution that is periodic with period 2π. You should also comment on how well the two methods are able to reproduce this periodic behaviour for long time intervals.

   (b) Check the numerical solutions you produced in part a) and report on how well they conserve the following quantities, which should remain constant:

   - Conservation of Energy

     $$\frac{(x')^2 \; + \; (y')^2}{2} \; - \; \frac{1}{r},$$

   - Conservation of Angular Momentum

     $$xy' \; - \; yx'.$$

   (c) Plot the "energy drift" and the "angular momentum drift" over time for both methods and different values of the error tolerance and comment on the results.

2. Assume you have been shown the following 'new' implicit formula for solving an IVP,

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{6}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \\ &\quad + \frac{2h}{3} f(x_i + \frac{h}{2}, \frac{y_i + y_{i+1}}{2} - \frac{h}{8}[f(x_{i+1}, y_{i+1}) - f(x_i, y_i)]). \end{aligned}$$

Show that this is in fact a Runge-Kutta formula by converting it to its standard RK tabular form. Determine the order of the formula. (Hint; After identifying the equivalent RK tableau, use the order conditions as identified in the reference Hairer et al. [1988] or Butcher [2003]. Note that the text discusses some of the necessary order conditions but does not identify all of them.)