CSC2302H

Assignment 2

October 12, 2010 University of Toronto **Due:** November 9, 2010

1. Scientists have recently discovered a stable orbit that arises in the simulation of the restricted three-body problem (where the orbits are planar). The bodies have equal mass (in our example we assume $m_1 = m_2 = m_3 = 1.0$) and, with the appropriate starting conditions, will follow the same figure-eight orbit as a periodic steady-state solution. Figure 1a shows the solution behaviour over several orbits by displaying the first spatial coordinate (as a function of x) over the interval of interest for each body. Figure 1b displays the "phase portrait" of the solution to this problem. This phase portrait is defined to be a plot of one spatial coordinate vs the other spatial coordinate for each of the three bodies. For the set of initial conditions in our example the solution vector is periodic and the orbits are identical (although out of phase) for each body.

   The two spatial coordinates of the j^{th} body are y_{1j}, y_{2j} for $j = 1, 2, 3$. Each of the six coordinates satisfy a second-order differential equation (which can be easily derived from Newtons law of motion applied to gravitational systems – see for example, Shamine[1994, pp. 90-95]):

$$y''_{ij} = \sum_{k=1,k\neq j}^{3} m_k\left(\frac{y_{ik} - y_{ij}}{d_{jk}^3}\right),$$

   where

$$d_{kj}^2 = \sum_{i=1}^{2}(y_{ik} - y_{ij})^2, \;\; k, j = 1, 2, 3.$$

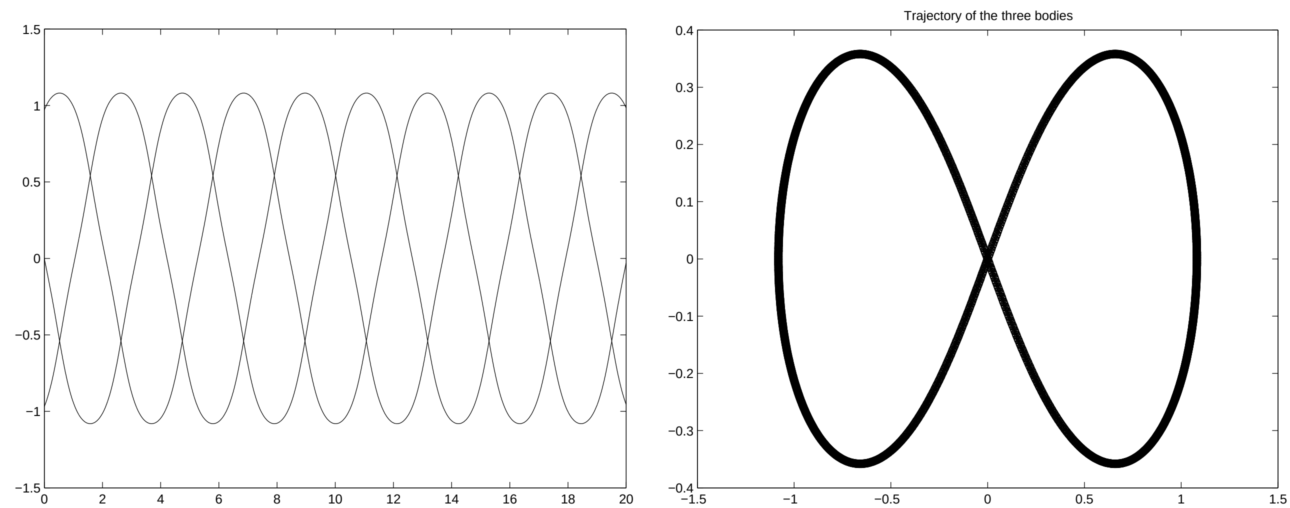
   When this system is re-written, as a first order system, the dimension of the problem is 12 and the initial conditions, at $x = 0$, are given by,

$$\begin{aligned}
y_{11} &= -0.97000436, & y'_{11} &= 0.466203685,\\
y_{21} &= 0.24308753, & y'_{21} &= 0.43236573,\\
y_{12} &= 0.0, & y'_{12} &= -0.93240737,\\
y_{22} &= 0.0, & y'_{22} &= -0.86473146,\\
y_{13} &= 0.97000436, & y'_{13} &= 0.466203685,\\
y_{23} &= -0.24308753, & y'_{23} &= 0.43236573.
\end{aligned}$$

   We are interested in this solution for $x \in [0, 40]$.

   (a) Use ode113 of MatLab to compute the solution with the given initial conditions and determine the period of the solution.

   (b) You are to determine, through some numerical experiments, whether the stable periodic solution observed in this example is sensitive to the assumption of constant mass. In particular, is the solution still periodic and if so, how does the period depend on ρ when $m_1 = m_3 = 1.0$ and $m_2 = 1.0 + \rho$. Note that the given initial conditions are on the periodic solution for the constant mass case. When this assumption is relaxed, these initial values are not likely to be on the periodic solution (if one exists) and it may take several multiples of the period before convergence to a periodic solution is observed.



(a) Solution Plot

(b) Phase Portrait

Figure 1: The approximate solution produced by ode45 for the Three Body problem

To help you get started, the MatLab script for specifying the vector of initial conditions and for evaluating the equivalent ode (after conversion to a first order system) is given by:

```
% specify the initial conditions Y0
   x00 = [-0.97000436; 0.24308753]; xp0 = [0.466203685;0.43236573];
   x10 = [0;0];                     xp1 = [-0.93240737;-0.86473146];
   x20 = [ 0.97000436;-0.24308753]; xp2 = [0.466203685;0.43236573];
   i0=[x00; xp0; x10; xp1; x20; xp2];
%---------The Function f.m (for three body problem-------------------
function Ydot = bodyf(t,Y,p1)
m0=1.0; m1=1.0; m2=1.0;   %masses of the three bodies
x0 = Y(1:2); x1 = Y(5:6); x2 = Y(9:10);
d0 = (x2-x1)/norm(x2-x1)^3;
d1 = (x0-x2)/norm(x0-x2)^3;
d2 = (x1-x0)/norm(x1-x0)^3;
Ydot(1:2) = Y(3:4);
Ydot(5:6) = Y(7:8);
Ydot(9:10) = Y(11:12);
Ydot(3:4) = m1*d2 - m2*d1;
Ydot(7:8) = m2*d0 - m0*d2;
Ydot(11:12) = m0*d1 - m1*d0;
Ydot=Ydot';
%---------------------------------------------------
```

2. Assume you have been shown the following 'new' implicit formula for solving an IVP,

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{6}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \\ &\quad + \frac{2h}{3} f(x_i + \frac{h}{2}, \frac{y_i + y_{i+1}}{2} - \frac{h}{8}[f(x_{i+1}, y_{i+1}) - f(x_i, y_i)]). \end{aligned}$$

Show that this is in fact a Runge-Kutta formula by converting it to its standard RK tabular form. Determine the order and region of absolute stability of the formula. (Hint; After identifying the equivalent RK tableau, use the order conditions as identified in the reference Hairer et al. [1988] or Butcher [2003]. Note that the text discusses some of the necessary order conditions but does not identify all of them.)