CSC2302H

Assignment 1

September 28, 2010 University of Toronto **Due:** October 12, 2010

For $\alpha = .08$, there are five stable periodic solutions of the forced Duffing Oscillator defined by the second order scalar ODE,

$$y'' + \alpha y' + y^3 = .2\cos(t).$$

These stable trajectories or 'attractors' can be characterized by the value of the period T and the vector of initial conditions, $A \equiv (y(0), y'(0))$. To two significant figures, the corresponding five values of A are $(-0.21, 0.02)$, $(1.05, 0.77)$, $(-0.67, 0.02)$, $(-0.46, 0.30)$, $(-0.43, 0.12)$ and the respective periods are $T = 2\pi, 2\pi, 4\pi, 4\pi, 6\pi$.

When we start this system at $t = 0$ from arbitrary initial conditions there will usually be a transient region before the solution approaches one of these five attractors. It is often of interest to associate with each attractor the ensemble of initial conditions (u, v) such that, starting with $y(0) = u, y'(0) = v$, the solution of the resulting initial value problem will settle to the attractor. This set is called the basin of attraction associated with the attractor. The union, over all attractors, of the associated basins of attraction will equal the entire (u, v) plane.

In this assignment you are to use the initial value routine, DVERK (and its associated continuous piecewise polynomial approximate solution, $S(x)$ – the routine INTRP) to perform a numerical investigation of the qualitative behaviour of solutions to this differential equation (for two values of α). The source code for the version of DVERK/INTRP you are to use can be downloaded from the course web-page as can the associated documentation (which is a generic documentation for all routines in the DVERK 'family' of numerical methods). You should note that, to apply DVERK to this scalar second-order problem, you must first convert it to an equivalent first-order problem of dimension 2. In order to access the piecewise polynomial associated with DVERK you must use interrupt # 2 (C(9) set to 1) and monitor each attempted step of the computation. On successful steps from x_i to $x_{i+1} = x_i + H$ (when IND has the value 5), you can evaluate the associated piecewise polynomial $S(x)$ to approximate to $y(x)$ for any $x \in (x_i, x_{i+1})$ by invoking the subroutine INTRP.

1. Note that the values for A specified above are only accurate to two significant digits and you should first make some preliminary runs in order to determine a more accurate value for each attractor. Using these more accurate values and the knowledge of the period of each attractor, you are to develop an efficient scheme to determine, for a given value of (u, v), which of the five basins of attraction (u, v) belongs to. [Warning: – when the period $T > 2\pi$, the same attractor can be characterized by different values of A. That is, the attractor can be characterized by $A = (y(0), y'(0))$ or by $\bar{A} = (y(2\pi), y'(2\pi))$.]

2. Using the program you have developed and a grid covering the region of the (u, v) plane, $-2 \le u \le 2$ and $-2 \le v \le 2$, show how this region is partitioned by the five basins of attraction. If you know that the prescribed initial conditions are equally likely to be anywhere in the range $-2 \le u \le 2$ and $-2 \le v \le 2$, estimate the relative likelihood that the system will converge to each of the 5 attractors.

3. Repeat the above investigation for $\alpha = .075$. In this case you will have to first determine the number of stable periodic solutions, the corresponding values of A, and the respective periods.