Interpolation and Approximation

- **The Basic Problem:**
  Approximate a continuous function $f(x)$, by a polynomial $p(x)$, over $[a, b]$.
  - $f(x)$ may only be known in tabular form.
  - $f(x)$ may be expensive to compute.
- **Definition:**
  A polynomial $p(x)$ interpolates $f(x)$ at the nodes $x_0, x_1, \cdots x_n$ if $p(x_i) = f(x_i)$ for $i = 0, 1, \cdots n$.
  (Intuitively if $f(x)$ and $p(x)$ agree at the x_i then they should be 'close' at nearby points.)

Key Theorem

Given distinct nodes $x_0, x_1, \cdots x_n$ and arbitrary $f_0, f_1, \cdots f_n$, there is a unique polynomial $p_n(x)$ of degree at most n that interpolates $f(x)$ at $x_0, x_1, \cdots x_n$.

Proof:

- Existence: (constructive)
  Let
  $$P_n(x) = a_0 + a_1(x - x_0) + \cdots a_n(x - x_0) \cdots (x - x_{n-1}),$$
  For any choice of $a_0, a_1, \cdots a_n, P_n(x)$ will be of degree at most n. We will choose the a_i to ensure that $P_n(x_i) = f_i$. This results in a system of $n + 1$ linear equations in the $n + 1$ unknowns, $a_0, a_1, \cdots a_n$. The i^{th} equation is:
  $$a_0 + a_1(x_i - x_0) + \cdots a_n(x_i - x_0) \cdots (x_i - x_{n-1}) = f_i,$$

Proof (cont)

In matrix form then, $B\underline{a} = \underline{f}$, where $\underline{a}^T = [a_0, a_1, \cdots a_n]^T, \underline{f} = [f_0, f_1, \cdots f_n]^T$, and B is the matrix,

$$b_{ij} = \begin{cases} 1 & \text{for } j = 0;, \\ (x_i - x_0)\cdots(x_i - x_{j-1}) & \text{for } j = 1, \cdots n;. \end{cases}$$

- B is lower triangular since b_{ij} contains a factor $(x_i - x_i)$ for $j > i$.
- $b_{ii} \neq 0$ since $x_0, x_1, \cdots x_n$ are distinct. This implies B is nonsingular and there exists a unique solution $\underline{a}$. Solving this triangular linear system by forward substitution:

$$\begin{aligned} a_0 &= f_0, \\ a_1 &= (f_1 - a_0)/(x_1 - x_0), \\ &\vdots \\ a_n &= [f_n - (a_0 + \cdots a_{n-1}(x_n - x_0)\cdots(x_n - x_{n-2}))] \\ &\quad /[(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})]. \end{aligned}$$

Two Key Observations

- The first $r+1$ terms, $a_0, a_1, \cdots a_r$, determine a polynomial of degree at most r that interpolates $f(x)$ at $x_0, x_1, \cdots x_r$.
- The resulting $P_n(x)$ satisfies the interpolation conditions and the coefficients $a_0, a_1, \cdots a_n$ define the Newton Form of this polynomial. (The Newton Form depends on the nodes and their order.)

Proof of Uniqueness

- Let $q_n(x)$ be a polynomial of degree at most n such that $q_n(x_i) = f_i$ for $i = 0, 1, \cdots n$. Then with $r(x) \equiv P_n(x) - q_n(x)$ we observe that $r(x)$ is a polynomial of degree at most n such that $r(x_i) = 0$ for $i = 0, 1, \cdots n$.

  Then, by Rolle's theorem, $r'(x)$ has n distinct zeros, $r''(x)$ has $n - 1$ distinct zeros, ... and $r^n(x)$ has one zero in the interval containing the nodes.

  But this is impossible unless $r(x) \equiv 0$, in which case $q_n(x)$ must equal $P_n(x)$.

Representation of $P_n(x)$

The interpolating polynomial $P_n(x)$ is unique but it can be represented in different ways. Consider the following representations of $P_n(x)$:

- Monomials (powers of x or $(x - w)$):

$$P_n(x) = c_0 + c_1 x + \cdots c_n x^n,$$

(In this case $P_n(x)$ is represented by the coefficients $c_0, c_1, \cdots c_n$ and we do not need to know the nodes or their order to evaluate or use this polynomial.)

- Newton Basis (or divided differences):

$$P_n(x) = a_0 + a_1(x - x_0) + \cdots a_n(x - x_0) \cdots (x - x_{n-1}),$$

(In this case $P_n(x)$ is represented by the a_i's and one must know the nodes and their order to use this polynomial.)

Representation of $P_n(x)$ (cont)

- Lagrange Basis:
  We introduce the j^{th} Lagrange basis function $\ell_j(x)$ (associated with the nodes $[x_i]_{i=0}^n$) by

$$\ell_j(x) = \prod_{i=0, i \neq j}^{n} (x - x_i) / \prod_{i=0, i \neq j}^{n} (x_j - x_i) = \prod_{i=0, i \neq j}^{n} \frac{(x - x_i)}{(x_j - x_i)},$$

  for $j = 0, 1, \cdots n$.
  Then it is clear that $\ell_j(x)$ is a polynomial of degree n such that

$$\ell_j(x_i) = \begin{cases} 0 & \text{for } j = 0, 1, \cdots n; j \neq i, \\ 1 & \text{for } j = i. \end{cases}$$

Lagrange Basis (cont)

It is also clear that any linear combination of the $\ell_j(x)$ will be a polynomial of degree at most n. In particular,

$$P_n(x) = \sum_{j=0}^{n} f_j \ell_j(x),$$

since $P_n(x)$ is unique and the polynomial $\sum_{j=o}^{n} f_j \ell_j(x)$ satisfies,

$$\sum_{j=0}^{n} f_j \ell_j(x_i) = f_i, \text{ for } i = 0, 1, \cdots n.$$

(In this case $P_n(x)$ is represented by the function values, $f_0, f_1, \cdots f_n$ and we must know the nodes to use $P_n(x)$.)

The particular choice of what representation to use will depend on the application. Often we will use the Lagrange form in our analysis but other forms in our implementations.

An Example

Consider the unique quadratic defined by the three interpolating conditions, $P_2(-1) = 7, P_2(0) = 2,$ and $P_2(1) = 1$ (that is, $x_0 = -1, x_1 = 0, x_2 = 1$ and $f_0 = 7, f_1 = 2, f_2 = 1$). The Lagrange basis is,

$$\begin{aligned}
\ell_0(x) &= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x^2 - x}{2}, \\
\ell_1(x) &= \frac{(x+1)(x-1)}{(0+1)(0-1)} = \frac{x^2 - 1}{-1} = 1 - x^2, \\
\ell_2(x) &= \frac{(x+1)x}{(1+1)(1-0)} = \frac{x^2 + x}{2}.
\end{aligned}$$

An Example (cont)

In Lagrange form $P_2(x)$ is

$$\begin{aligned} P_2(x) &= f_0\ell_0(x) + f_1\ell_1(x) + f_2\ell_2(x), \\ &= 7(\frac{x^2 - x}{2}) + 2(1 - x^2) + \frac{x^2 + x}{2}. \\ (&= 2x^2 - 3x + 2 \quad \text{(in monomial form)}) \end{aligned}$$

Divided Differences

- Definition: The divided difference, $f[x_0 x_1 \cdots x_i]$ is defined to be a_i the $(i+1)^{st}$ coefficient of the interpolating polynomial $P_n(x)$ written in Newton form.

$$P_n(x) = a_0 + a_1(x - x_0) + \cdots a_n(x - x_0)\cdots(x - x_{n-1}).$$

- Recall that the first $r+1$ coefficients of the Newton form of $P_n(x)$ determine a polynomial of degree at most r that interpolates $f(x)$ at the nodes $(x_0, x_1, \cdots x_r)$.

Properties of Divided Differences

- $f[x_0 x_1 \cdots x_i] = f[x_{j_0} x_{j_1} \cdots x_{j_i}]$,
  where $j_0, j_1, \cdots j_i$ is a permutation of $0, 1, \cdots i$.

- $f[x_0 x_1 \cdots x_i] = \frac{f[x_0 x_1 \cdots x_{i-1}] - f[x_1 x_2 \cdots x_i]}{x_0 - x_i}$.

Proof of First Property

The polynomial of degree at most i interpolating $f(x)$ at $(x_0, x_1, \cdots x_i)$ is

$$P_i(x) = f[x_0] + f[x_0 x_1](x - x_0) + \cdots f[x_0 x_1 \cdots x_i](x - x_0)\cdots(x - x_{i-1}),$$

Note that the coefficient of x^i in this polynomial is $f[x_0 x_1 \cdots x_i]$. Similarly the polynomial $\bar{P}_i(x)$ interpolating $f(x)$ at $(x_{j_0}, x_{j_1}, \cdots x_{j_i})$ is

$$\bar{P}_i(x) = f[x_{j_0}] + f[x_{j_0} x_{j_1}](x - x_{j_0}) + \cdots f[x_{j_0} x_{j_1} \cdots x_{j_i}](x - x_{j_0})\cdots(x - x_{j_{i-1}}).$$

Now the sets $[x_0, x_1, \cdots x_i]$ and $[x_{j_0}, x_{j_1}, \cdots x_{j_i}]$ are identical (only the order may be different) so by uniqueness we have $P_i(x) = \bar{P}_i(x)$. In particular the respective coefficients of x^i must agree and we have,

$$f[x_0 x_1 \cdots x_i] = f[x_{j_0} x_{j_1} \cdots x_{j_i}].$$

Proof of Second Property

Consider the orderings $(x_1, x_2, \cdots x_i, x_0)$ and $(x_0, x_1, \cdots x_i)$. Writing $P_i(x)$ using the two different Newton Forms (corresponding to these different orderings) we have,

$$\begin{aligned} P_i(x) &= f[x_0] + f[x_0 x_1](x - x_0) + \cdots f[x_0 x_1 \cdots x_i](x - x_0) \cdots (x - x_{i-1}) \\ &= f[x_1] + f[x_1 x_2](x - x_1) + \cdots f[x_1 x_2 \cdots x_i x_0](x - x_1) \cdots (x - x_i). \end{aligned}$$

Multiplying the first equation by $(x - x_i)$, the second equation by $(x - x_0)$ and subtracting we obtain,

$$(x - x_i)P_i(x) - (x - x_0)P_i(x) = (x_0 - x_i)P_i(x),$$

or ...

Proof (cont)

$$\begin{aligned}(x_0 - x_i)P_i(x) \quad = \quad & f[x_0](x - x_i) - f[x_1](x - x_0) \\ & \vdots \\ & + f[x_0 x_1 \cdots x_{i-1}](x - x_0)\cdots(x - x_{i-2})(x - x_i) \\ & \quad - f[x_1 x_2 \cdots x_i](x - x_0)\cdots(x - x_{i-1}) \\ & + f[x_0 x_1 \cdots x_i](x - x_0)\cdots(x - x_i) \\ & \quad - f[x_1 x_2 \cdots x_i x_0](x - x_0)\cdots(x - x_i).\end{aligned}$$

But the last term in this expansion vanishes and this implies the coefficient of x^i in the polynomial $(x_0 - x_i)P_i(x)$ is $f[x_0 x_1 \cdots x_{i-1}] - f[x_1 x_2 \cdots x_i]$. We know this coefficient is $(x_0 - x_i)f[x_0 x_1 \cdots x_i]$ so we can conclude,

$$f[x_0 x_1 \cdots x_i] = \frac{f[x_0 x_1 \cdots x_{i-1}] - f[x_1 x_2 \cdots x_i]}{x_0 - x_i}.$$

Error in Interpolation

Let $E_n(x) = f(x) - P_n(x)$. To investigate the behaviour of $E_n(x)$ consider fixing x and determine the polynomial $p_{n+1}(z)$ of degree at most $n+1$ (in z), interpolating $f(z)$ at the $n+2$ nodes $(x_0, x_1, \cdots x_n, x)$.

$$\begin{aligned} p_{n+1}(z) &= f[x_0] + f[x_0 x_1](z - x_0) + \cdots \\ &\quad f[x_0 x_1 \cdots x_n x](z - x_0)\cdots(z - x_n) \\ &= P_n(z) + f[x_0 x_1 \cdots x_n x](z - x_0)\cdots(z - x_n). \end{aligned}$$

Evaluating this expression at $z = x$ and using the fact that $p_{n+1}(x) = f(x)$ we have,

$$\begin{aligned} E_n(x) &= f(x) - P_n(x) = p_{n+1}(x) - P_n(x), \\ &= f[x_0 x_1 \cdots x_n x](x - x_0)\cdots(x - x_n). \end{aligned}$$

Error Expression

We have shown,

$$E_n(x) = f[x_0 x_1 \cdots x_n x](x - x_0) \cdots (x - x_n)$$

Therefore if $f[x_0 x_1 \cdots x_n x]$ (as a function of x) is 'slowly varying', then we can estimate $E_n(x)$ by

$$est_n(x) = \prod_{i=0}^{n}(x - x_i) f[x_0 x_1 \cdots x_n x_{n+1}]$$

for some x_{n+1}.