CSCC51H

Mid-term Examination

February 29, 2012 Scarborough Campus

Time: 50 minutes.

Answer both questions showing all work in the examination booklet.

The questions are equally weighted.

Aids allowed: Approved calculator and one 8.5" x 11" page of handwritten notes.

1. (Hermite Interpolation):
   The following tabulated values of $f(x)$ and $f'(x)$ are given.

x_i	$f(x_i)$	$f'(x_i)$
0	1.00	0.00
1	0.368	-0.736
2	0.018	-0.073

   (a) Determine the Hermite cubic interpolant, $p_3(x)$, associated with the two data points $x_0 = 0$ and $x_2 = 2$.

   (b) Evaluate this cubic and its derivative at $x = 1$ and determine the resuting approximation errors using the second row of the above table to define the "exact" values of $f(1)$, $f'(1)$.

2. (Numerical integration):
   If $f(x) \in C^1[a, b]$. then the *total variation* (TV) of $f(x)$ on the interval $[a, b]$ is defined to be,

$$TV = \int_a^b (1 + (\frac{df}{dx})^2)^{1/2} dx.$$

   (a) For the function tabulated in problem 1, suggest an algorithm that could be used to compute an approximation to the TV of $f(x)$ on $[0, 2]$. Your algorithm should use only the data given in the table of problem 1.

   (b) Apply your algorithm and compute an approximation to the TV of $f(x)$ on $[0, 2]$ for this function.

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1.

x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
0	1			
0	1	0		
2	.018	$\frac{.018-1}{2} = -.491$	$\frac{-.491}{2} = -.245$	
2	.018	$-.073$	$\frac{-.073+.491}{2} = .209$	$\frac{.209+.245}{2} = .227$

$$P_3(x) = 1 + 0\cdot x - .245\cdot x^2 + .227\, x^2(x-2)$$

$$= 1 - .699\, x^2 + .227\, x^3$$

$$P_3(1) = 1 - .699 + .227 = .528$$

$$P_3'(x) = -1.38\, x + .681\, x^2$$

$$p'(1) = -1.38 + .681 = -.699$$

$$e(1) = .368 - .528 = -.160$$

$$\hat{e}(1) = -.736 + .699 = -.037$$

2.

Let $g(x) = \left(1+(f'(x))^2\right)^{1/2}$

A straightforward, not optimal method would use

Composite Simpson rule applied to $g(x)$

$$TV \approx \frac{(b-a)}{6}\left(g(a) + 4\,g\left(\tfrac{a+b}{2}\right) + g(b)\right)$$

$$= \frac{1}{3}\left(g(0) + 4\,g(1) + g(2)\right)$$

but from the table:

$$g(0) = (1+0)^{1/2} = 1, \quad g(1) = \left(1+(.736)^2\right)^{1/2} = 1.24$$

$$g(2) = \left(1+(.073)^2\right)^{1/2} = 1.00$$

$$\therefore TV \approx \frac{1}{3}\left(1 + 4\times 1.24 + 1.00\right)$$

$$= \frac{1}{3}\times 6.96 = 2.32$$

The above answer would warrant 9/10. A better answer would make use of all the data of the Table to produce a more accurate approximation to $f(x)$ on $[0,2]$

eg) $p_5(x) \approx f(x)$ on $[0,2]$ where $p_5(x)$ interpolates all the data. ~~the~~ Then define $\hat{g}(x) = \left(1+(p_5'(x))^2\right)^{1/2}$ and

Approx $\int_0^2 (1+(\hat{g}'(x))^2)^{1/2}\,dx$