CSCC51H

Final Examination

April 21, 2011 University of Toronto at Scarborough

Time: 3 Hours.

Answer all four questions showing all work in the examination booklet(s).

The questions are equally weighted.

Aids allowed: Approved calculator and one 8.5 x 11 page of handwritten notes.

1. (A quadrature problem):
You are given a function $f(x) \in C^8[a,b]$ with $f(a) \geq 0$. Describe in detail an efficient technique you could use to provide an accurate approximation to,
$$\int_a^b |f(x)|\, dx.$$
   (a) What is the order of accuracy of your technique if it is known that $f(x) \geq 0$ for $x \in [a,b]$.

   (b) Note that, if $f(x)$ changes sign on $[a,b]$, then the integrand will not be differentiable. What would the order of accuracy be for your technique in this case. How could the order be improved on such problems?

   (c) How could you estimate the error in the case of your improved technique when $f(x)$ changes sign on $[a,b]$?

2. (Estimating the Maximum Speed from Tabulated Data):
In an atempt to determine the maximum speed of vehicles, sensors have been placed on a highway located 100 meters apart. The sensors record accurate values of the time a vehicle passes by and the velocity of the vehicle as it passes. For each vehicle it records the measured values in a table. You are to design an effective technique to estimate, from these tables, the maximum speed the vehicle reached as it passed over the 100 meter section of highway. Note that the recorded velocity, v, is measured in meters per second and the time, t, is measured in seconds.

   (a) Describe in detail your technique and justify why you feel it is better than approximating the maximum speed by taking the maximum of the two recorded velocity values.

   (b) Apply your technique to a typical table of recorded values given by $v = 33.3$ and $t = 0.0$ at sensor No.1, and $v = 22.2$ and $t = 3.0$ at sensor No.2. Also compute the average speed of this vehicle and discuss why, in this case, your estimated maximum speed is more believable than the maximum of the two recorded values.

   (c) In order to justify your approach, what assumptions must be made on the accuracy of the recorded values and the "smoothness" of the velocity function.

3. (Higher order ODEs):
Assume you have a numerical method available for solving first order systems of IVPs and you are given a second order system of IVPs,

$$y'' = y^3 + u^3, \quad u' = y + u, \quad \text{for } x \in [0, 1],$$

with $y(0) = y'(0) = 1, \ u(0) = 2$.

(a) Show how to convert this problem to an equivalent first order system of IVPs, of dimension 3,

$$z' = \bar{f}(x, z), \quad z(0) = z_0.$$

(b) In this application, assume it is important to approximate the function,

$$h(t) = \int_0^t (y'(s) + u'(s))^2 ds,$$

for $t \in [0, 1]$. Describe how you could reliably approximate the function $h(t)$ for $t \in [0, 1]$ using any of the quadrature or ODE methods we have discussed in this course.

4. (Properties of Runge-Kutta Formulas):
A general s-stage, order p, Runge-Kutta formula can be represented by the tableau:

$$\begin{array}{c|ccccc}
- & - & & & & \\
\alpha_2 & \beta_{21} & - & & & \\
\vdots & \vdots & & & & \\
\alpha_s & \beta_{s1} & \cdots & & \beta_{s-1,s} & - \\
\hline
 & \omega_1 & \omega_2 & \cdots & & \omega_s
\end{array}$$

Consider applying this formula to the initial value problem, $y' = \lambda y, \ \ y(0) = y_0$ with a constant stepsize, h.

(a) For the special case of the Modified Euler formula (where $s = 2$ and $\alpha_2 = \beta_{21} = 1, \omega_1 = \omega_2 = 1/2$) derive explicit expressions for k_1 and k_2 for the first step.

(b) Show that for the general s-stage formula, for $r = 1, 2, \ldots s$, the stages k_r satisfy, on the first step,

$$k_r = \lambda p_{r-1}(h\lambda)y_0,$$

where $p_{r-1}(z)$ is a polynomial (in z) of degree at most $r - 1$. [Hint use induction on r.]

(c) Using the result of part (b), prove that any order s formula of this type must produce the same approximation to y_1. In particular all 2-stage, second order formulas (such as the family of such formulas derived in class) must produce the same approximate solutions y_i for all steps when applied with a constant h.

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1. a) Any higher order ≥ 2 method is acceptable

eg - Composite Trapezoidal has order 2 $(O(h^2))$

(5) - Composite Simpson rule order 4 $(O(h^4))$

b) $O(h)$ in bisection

- can be expressed by finding $x_i \ni f(x_i) = 0$

(10)

and all case x_{i-1} x_i were called for (a)

c) $err = \sum_{\substack{i=1 \\ \text{subinterval } x_{i-1}, x_i}}^{K} |err_i|$

(5)

2 Approximate $d(t)$ using cubic Hermite polynomial

t_i	$d_i[t_i]$	$v_i = d[t_{i-1}, t_i]$	$d[t_{i-2}, t_{i-1}, t_i]$	$d[t_{i-3}, t_{i-2}, t_{i-1}, t_i]$
0	$\boxed{0}$			
0	0	$\boxed{33.3}$		
3	100	$\frac{100-0}{3-0} = 33.3$	$\frac{33.3-33.3}{3} = \boxed{0}$	$\frac{-3.7}{3} = \boxed{-1.23}$
3	100	22.2	$\frac{22.2-33.3}{3} = -3.7$	

$$\therefore\ d(t) = 0 + 33.3t + 0\,t^2 - 1.23\,t^2(t-3)$$

$$= 33.3t + 3.69t^2 - 1.23t^3$$

check $d(0) = 0$, $d'(0) = 33.3$, $d(3) = 100 - 27\times 1.23 + 3.69\times 9 = 100$

$d'(3) = 33.3 - 27\times 1.23 + 3.96\times 6 = 22.2$

a) Any reasonable justification + technique

(8)

(7) b) max ~~spd~~ speed occurs at $\bar{t}$ where $v' = d''(t) = 0$ or

$\bar{t} = 0$ or $\bar{t} = 3$

but $d''(t) = 7.38 - 7.38t$

$= 0$ when $\bar{t} = 1$

and $d'(1) = 33.3 + 7.38 - 3.69 = 37$ m/sec

AVG = 33.3

(5) c) At least 3 ~~sig~~ sig figures in velocity + clock

accurate to 3 sig accuracy

and velocities are diff or accelerations are

continuous

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a)

(5) $z = \begin{bmatrix} y \\ y' \\ u \end{bmatrix} \Rightarrow z' = \begin{bmatrix} y'(x) \\ y''(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} z_2(x) \\ z_1^3(x) + z_3^3(x) \\ z_1(x) + z_3(x) \end{bmatrix} = \bar{f}(x, z), \quad z_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

1) one solution introduces $z_4(x) \overset{h(x) \equiv}{=} \int_0^x (y'(s) + u'(s))^2 \, ds$

then $z_4'(x) = (y'(x) + u'(x))^2$

$= [z_2(x) + (z_1(x) + z_3(x))]^2$

(15) with $z_4(0) = 0$

$\therefore \; \bar{z}(x) \in R^4 \equiv \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$ satisfies the IVP

$\bar{z}'(x) = \begin{bmatrix} \bar{z}_2(x) \\ \bar{z}_1^3(x) + \bar{z}_3^3(x) \\ \bar{z}_1(x) + \bar{z}_3(x) \\ (\bar{z}_2(x) + \bar{z}_1(x) + \bar{z}_3(x))^2 \end{bmatrix} = \hat{f}(x, \bar{z}), \quad \bar{z}(0) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

can apply any ODE solver and plot $\bar{z}_4(x)$

4) a) $k_1 = f(x_0 y_0) = \lambda y_0$

(5) $k_2 = f(x_0 + h, y_0 + h k_1) = \lambda[y_0 + h\lambda y_0] = \lambda(1 + h\lambda) y_0$

b) induction hypothesis $k_r = \lambda p_{r-1}(h\lambda) y_0$ for $r = 1, 2 \cdots$

true for base case $r = 1$ since as in a)

k_1 always $= \lambda y_0 = \lambda p_0(h\lambda) y_0$

where p_0 is a polynomial of degree $0 \equiv 1$

assume for $r > 1$ $r \le n$ and ind hyp is satisfied

10

for $i < r$, ie for $i < r$, $k_i = \lambda p_{i-1}(h\lambda) y_0$

then $k_r = f(x_0 + \alpha_r h, y_0 + h \sum_{j=1}^{r-1} \beta_{rj} k_j) = \lambda(y_0 + h \sum_{j=1}^{r-1} k_j)$

but for $j < r$ $k_j = \lambda p_{j-1}(h\lambda) y_0$

so $k_r = \lambda(y_0 + h \sum_{j=1}^{r-1} \lambda p_{j-1}(h\lambda) y_0)$

$= \lambda \underbrace{\left(1 + \sum_{j=1}^{r-1} (h\lambda) p_{j-1}(h\lambda)\right)} y_0$

$= \lambda \quad p_j(h\lambda) \quad y_0$

QED

(5) c) True local solution $y(x_1) = e^{h\lambda} y_0$

$y_1 = y_0 + h \sum_{r=1}^{\hat{s}} \omega_r k_r = y_0 + \sum_{r=1}^{s} (h\lambda) p_{r-1}(h\lambda) y_0$

$= (1 + q_s(h\lambda)) y_0 = P_s(h\lambda) y_0$

$LE = O(h^{p+1}) = (e^{h\lambda} - P_s(h\lambda)) = O(h^{s+1})$

$\therefore LE = \left(e^{h\lambda} - p_s(h\lambda)\right) y_0 = O(h^{s+1})$

which means that $p_s(h\lambda)$ must

equal $\sum_{j=0}^{s} \frac{(h\lambda)^j}{j!}$!

Note this question is not easy and

I did not expect anyone to give a

complete solution. W. E.