CSCC51H

Assignment 4

March 12, 2012 University of Toronto **Due:** April 4, 2012

(Reference Kahaner, Moler and Nash: Problem 8-6, pp. 333-334, and text problem 9.8, pp. 420-421)

The following differential equations describe the motion of a body in orbit about two much heavier bodies. An example is an Apollo capsule in an earth-moon orbit. The coordinate system is a little tricky. The three bodies determine a plane in space and a two-dimensional Cartesian coordinate system in this plane. The origin is at the center of mass of the two heavier bodies, the x - axis is the line through these two bodies and the distance between them is the unit length. Thus, if μ is the ratio of the mass of the moon to that of the earth, then the moon and earth are located at coordinates $(1-\mu, 0)$ and $(-\mu, 0)$, respectively, and the coordinate system moves as the moon rotates about the earth. The third body, the Apollo capsule, is assumed to have a mass that is negligible compared to the other two, and its position as a function of time is $(x(t), y(t))$. The equations are derived from Newton's law of motion and the inverse square law for gravitation. The first derivatives in the equation come from the rotating coordinate system and from a friction term, which is assumed to be proportional to the velocity with proportionality constant f:

$$\begin{aligned} x'' &= 2y' + x - \frac{\tilde{\mu}(x+\mu)}{r_1^3} - \frac{\mu(x-\tilde{\mu})}{r_2^3} - fx' \\ y'' &= -2x' + y - \frac{\tilde{\mu}y}{r_1^3} - \frac{\mu y}{r_2^3} - fy' \end{aligned}$$

with

$$\mu = \frac{1}{82.45}, \;\; \tilde{\mu} = 1-\mu, \;\; r_1^2 = (x+\mu)^2 + y^2, \;\; r_2^2 = (x-\tilde{\mu})^2 + y^2.$$

Although a great deal is known about these equations, it is not possible to find a closed form solution. One interesting class of problems involves the investigation of periodic solutions in the absence of friction. It is known that the initial conditions,

$$x(0) = 1.2, \;\; x'(0) = 0, \;\; y(0) = 0, \;\; y'(0) = -1.04935751,$$

lead to a solution which is periodic with period $T = 6.19216933$, when $f = 0$. This orbit starts with the Apollo capsule on the far side of the moon with an altitude of 0.2 times the earth-moon distance and a prescribed initial velocity. The resulting orbit brings Apollo in close to the earth, out in a big loop on the opposite side of earth from the moon, back in close to the earth again, and finally back to its original position and velocity on the far side of the moon.

1. Use ode113 or ode45 (from MatLab) to compute the solution with the given initial conditions. Plot this orbit and verify that the solution is periodic with the given period. Using the interpolant provided with the code, determine how close Apollo comes to the surface of the earth in this orbit

   In the description above, distances are measured from the centers of the earth and moon. Assume that the moon is $238,000$ miles from the earth and that the earth is a sphere with radius $4,000$ miles. (Note that the origin of the coordinate system is within this sphere but not at its center.)

2. When $f = 1$, with the same initial conditions as in (1.), determine an approximate solution from $t = 0$ to $t = 5$. Plot the phase plane of the solution. That is, plot $x(t)$ versus $y(t)$. In this case, the Apollo capsule is 'captured' by the earth and eventually crashes.

3. Repeat the computations in (2.) with $f = 0.1$. By looking at the phase plane, can you guess what is happening? This will be easier if you repeat the calculation for a longer time, say to $t = 8$.

4. Discuss how you could determine (using Matlab) whether a given set of initial conditions leads to a periodic solution for this ODE. Note that, for 1., you verified that a prescribed set of initial conditions did produce a periodic solution with a known period, T. For this question, you are asked to *discuss* how you could determine whether a prescribed set of initial conditions does produce a periodic solution for some period, T.