Enumeration

September-12-11 1:36 PM

Geometric Series Expansion

$Q = 1 + z + z^2 + z^3 + \cdots$
$zQ = z + z^2 + z^3 + z^4 + \cdots$
$Q - zQ = 1$
$\therefore Q = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$

Example

Let a_n be the number of subset of $\{1, 2, \dots, n\}$ that don't contain two consecutive numbers.
Determine for all $n \geq 0$

n	subsets	a_n
0	$\emptyset$	1
1	$\emptyset, \{1\}$	2
2	$\emptyset, \{1\}, \{2\}$	3
3	$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}$	5

Let A_n be the collection of all such subsets of $\{1, 2, \dots, n\}$
Let B_n be the collection of these sets $S \in A_n$ for which $n \in S$
Then $A_n = A_{n-1} \cup B_n$ is a disjoint union of subsets.
So $a_n = |A_n| = |A_{n-1}| + |B_n|$
The set B_n is in bijection with A_{n-2}

$$\begin{cases} S \in B_n \ corresponds\ to\ S \setminus \{n\} \\ T \in A_{n-2}\ corresponds\ to\ T \cup \{n\} \in B_n \end{cases}$$

Hence $|B_n| = |A_{n-2}| = a_{n-2}$
Hence $a_n = a_{n-1} + a_{n-2}\ for\ n \geq 2$

Fibonacci Numbers

$f_0 = 1, f_1 = 1, f_n = f_{n-1} + f_{n-2}\ for\ n \geq 2$
So for us, $a_n = f_{n+1} for\ n \geq 0$

Get a formula for f_n as a function of n.

Generating Function

$$F = F(x) = \sum_{n=0}^{\infty} f_n x^n$$

From the initial conditions and the recurrence we get the following:

$$F = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$
$$= 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n$$
$$= 1 + x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$
$$= 1 + x + \sum_{i=1}^{\infty} f_i x^{i+1} + \sum_{j=0}^{\infty} f_j x^{j+2}$$
$$= 1 + x + x(F-1) + x^2(F)$$

$Hence$

$$F = 1 + xF + x^2F$$
$$F(x) = \sum_{n=2}^{\infty} f_n x^n = \frac{1}{1-x-x^2}$$

Now get expression for individual terms

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$$
$$x = \frac{1}{t} \Rightarrow t^2 - t - 1 = (t-\alpha)(t-\beta)$$
$$\alpha, \beta = \frac{1 \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2} = \frac{(1 \pm \sqrt{5})}{2}$$

By partial fractions $\exists A, B \in \mathbb{C}\ such\ that$

$$\frac{1}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$$\sum_{n=0}^{\infty} f_n x^n = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A \sum_{n=0}^{\infty} \alpha^n x^n + B \sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n) x^n$$

So

$$f_n = A\alpha^n + B\beta^n\ \forall n \geq 0$$

Initial Conditions

$$f_0 = 1 = A + B$$
$$f_1 = 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{(1-\sqrt{5})}{2}\right)$$

Solve for A, B

$$f_1 = 1 = \frac{A+B}{2} + \frac{(A-B)\sqrt{5}}{2}$$
$$2 = \left(1+\sqrt{5}\right)A + \left(1-\sqrt{5}\right)B$$
$$B = 1 - A$$

$$2 = \left(1+\sqrt{5}\right)A + \left(1-\sqrt{5}\right)(1-A) = A + \sqrt{5}A + 1 - \sqrt{5} - A + \sqrt{5}A = 1 - \sqrt{5} + 2\sqrt{5}A = 2$$
$$A = \frac{\sqrt{5}+1}{2\sqrt{5}}$$
$$B = 1 - A = \frac{2\sqrt{5} - 1 - \sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2\sqrt{5}}$$
$$f_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Generating Functions

September-14-11 1:28 PM

$$H = H(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{1 + x + 3x^2}{1 - 3x^2 - 2x^3}$$

Generating Function to Recurrence Relation

Convention: $h_n = 0 \; if \; n < 0$
Clear denominators

$$(1 - 3x^2 - 2x^3) \sum_{n=-\infty}^{\infty} h_n x^n = 1 + x + 3x^2$$

$$\sum_n h_n x^n - 3\sum_n h_n x^{n+2} - 2\sum_n h_n x^{n+3} = \sum_n h_n x^n - 3\sum_n h_{n-2} x^n - 2\sum_n h_{n-3} x^n$$

$$= \sum_n (h_n - 3h_{n-2} - h_{n-3}) x^n = 1 + x + 3x^2$$

$n = 0 \quad h_0 - 3h_{-2} - 2h_{-3} = 1 \Rightarrow h_0 = 1$
$n = 1 \quad h_1 = 1$
$n = 2 \quad h_2 - 3h_0 = 3 = 3 \Rightarrow h_2 = 6$
For all $n \geq 3$, $h_n - 3h_{n-2} - 2h_{n-3} = 0$
Hence
$h_0 = 1, h_1 = 1, h_2 = 6$
For $n \geq 3$: $h_n = 3h_{n-2} + h_{n-3}$

Recurrence Relation to Generating Function

$h_0 = 1, h_1 = 1, h_2 = 6$
$h_n = 3h_{n-2} + 2h_{n-3}$
$h_n = 0 \; if \; n < 0$

$$H = H(x) = \sum_n h_n x^n$$

$$1 + x + 6x^2 + \sum_{n=3}^{\infty} (3h_{n-2} + 2h_{n-3}) x^n = 1 + x + 6x^2 + \sum_{n=3}^{\infty} 3h_{n-2} x^n + \sum_{n=3}^{\infty} 2h_{n-2} x^n$$

$$= 1 + x + 6x^2 + \sum_{i=1}^{\infty} 3h_i x^{i+2} + \sum_{j=0}^{\infty} 2h_j x^{j+3}$$

$$H = 1 + x + 6x^2 + 3x^2(H - 1) + 2x^3 H$$

$$H(x) = \frac{1 + x + 3x^2}{1 - 3x^2 - 2x^3}$$

Generating Function to Coefficient Formula

Works only when $H(x) = \frac{P(x)}{Q(x)}$ with $\deg P < \deg Q$
Uses partial fraction expansion.

Factor the denominator, identifying **inverse roots.**

$1 - 3x^2 - 2x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x), \qquad \alpha, \beta, \gamma \in \mathbb{C}$

$t^3 - 3t - 2 = (t - \alpha)(t - \beta)(t - \gamma), \qquad where\; t = \frac{1}{x}$

$= (t + 1)(t^2 - t - 2) = (t + 1)^2(t - 2)$

Since $\deg(1 + x + 3x^2) < \deg(1 - 3x^2 - 2x^3)$ $\exists A, B, C \in \mathbb{C}$:

$$\frac{1 + x + 3x^2}{1 - 3x^2 - 2x^3} = \frac{A}{1 - 2x} + \frac{B}{1 + x} + \frac{C}{(1 + x)^2}$$

$1 + x + 3x^2 = A(1 + x)^2 + B(1 - 2x)(1 + x) + c(1 - 2x)$
$x = 0: 1 = A + B + C$
$x = -1: 3 = 0 + 0 + 3C \Rightarrow C = 1$
$x = \frac{1}{2}: \frac{9}{4} = \frac{9}{4}A + 0 + 0 \Rightarrow A = 1, B = -1$

$$\frac{1 + x + 3x^2}{1 - 3x^2 - 2x^3} = \frac{1}{1 - 2x} - \frac{1}{1 + x} + \frac{1}{(1 + x)^2}$$

Aside

$$\frac{1}{(1 - z)^2} = \frac{1}{1 - z} \times \frac{1}{1 - z} = \left(\sum_{i=0}^{\infty} z^i\right)\left(\sum_{i=0}^{\infty} z^i\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} z^{i+j}\right) = \sum_{n=0}^{\infty}\left(\sum_{\substack{i+j=n \\ i\geq 0, j\geq 0}} 1\right) z^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} 1\right) z^n = \sum_{n=0}^{\infty} (n + 1)\, z^n$$

$$\frac{1}{(1 + x)^2} = \sum_{n=0}^{\infty} (n + 1)(-x)^n$$

$$H = \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} (n + 1)(-1)^n x^n = \sum_{n=0}^{\infty} (2^n + n(-1)^n) x^n$$

Thus

$h_n = 2^n + n(-1)^n \; \forall n \geq 0$

Higher Powers

$$\frac{1}{(1-z)^3} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z^{i+j+k}$$

The coefficient is the number of solutions (i, j, k) to the equation $i + j + k$ where $i \geq 0, j \geq 0, k \geq 0 \in \mathbb{Z}$

Partial Fractions

September-16-11 1:31 PM

Partial Fractions

$$Q(x) = \prod_i (1-\alpha_i)^{k_i}$$

$$P(x) \text{ has degree} \leq \sum_i k_i$$

$$\frac{P(x)}{Q(x)} = \sum_i \sum_{j=1}^{k_i} \frac{A_{ij}}{(1-\alpha_i)^j}$$

Generating Function

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Multisets

Intuitively: sets with repeated elements
t "types" of element
each type can occur any number of times.
size of multiset = total # of occurrences of elements.

For each type of element $1 \leq i \leq t$ let m_i be the number of times that element of type i occurs in the multiset.

The size of the multiset is $m_1 + m_2 + \cdots + m_t$, where m is the multiplicity for element i
So the coefficient of x^3 $in \frac{1}{(1-x)^3}$ is

$$[x^3]\frac{1}{(1-x)^3} = 10$$

We can regard a multiset of size n with elements of t types as its sequence of multiplicities.

$(m_1, m_2, \dots, m_t) \in \mathbb{N}^t$ with $m_1 + m_2 + \cdots + m_t = n$

Fact

There are

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

k-element subsets of $\{1, 2, \dots, n\}$

Proposition

For $n \geq 0$ and $t \geq 1$ there are $\binom{n+t-1}{t-1}$ multisets of size n with elements of t types.

Partial Fractions Example

$\alpha, \beta, \gamma \in \mathbb{C}$ *distinct non* $-$ *zero*
$Q(x) = (1-\alpha x)(1-\beta x)^2(1-\gamma x)^3$
$P(x)$ has degree ≤ 5
By partial fractions
$\exists A, B, C, D, E, F \in \mathbb{C}$ such that

$$\frac{P(x)}{Q(x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} + \frac{C}{(1-\beta x)^2} + \frac{D}{1-\gamma x} + \frac{E}{(1-\gamma x)^2} + \frac{F}{(1-\gamma x)^3}$$

General Problem

$\frac{1}{(1-x)^t}$ as a power series in x.

$$t = 1: \frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$$

$$t = 2: \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

$$\frac{1}{(1-x)^t} = \left(\frac{1}{1-x}\right)^t = \left(\sum_{m=0}^{\infty} x^m\right)^t = \prod_{i=1}^{t}\left(\sum_{m_i=0}^{\infty} x^{m_i}\right) = \sum_{m_1}^{\infty}\sum_{m_2}^{\infty} \dots \sum_{m_t}^{\infty} x^{m_1+m_2+\cdots+m_t}$$

$$= \sum_{(m_1,m_2,\dots,m_t)\in\mathbb{N}^t} x^{m_1+m_2+\cdots+m_t}$$

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty}\left(\sum_{\substack{(m_1,m_2,\dots,m_t)\in\mathbb{N}^t \\ m_1+m_2+\cdots+m_t=n}} 1\right)x^n$$

The coefficient of x^n in $\frac{1}{(1-x)^t}$ is the number of n-tuples $(m_1, m_2, \dots, m_t) \in \mathbb{N}^t$ such that $\sum_{i=1}^{t} m_i = n$

Example of multisets

Multiset of size 3 with 3 types of elements: A, B, C
For each type of element $1 \leq i \leq t$ let m_i be the number of times that element of type I occurs in the multiset.

Multiset	m_1, m_2, m_3
A,A,A	3,0,0
A,A,B	2,1,0
A,A,C	2,0,1
A,B,B	1,2,0
A,B,C	1,1,1
A,C,C	1,0,2
B,B,B	0,3,0
B,B,C	0,2,1
B,C,C	0,1,2
C,C,C	0,0,3

Proof of Proposition

Establish a bijection between the set of t-type multisets of size n and the set of $(t-1)$-element subsets of $\{1, 2, \dots, n+t-1\}$

Informally

Write a sequence of $n + t - 1$ spaces.
Example: $n = 7, t = 4$
_ _ _ _ _ _ _ _ _ _
Cross out $t - 1$ of those spaces. Count empty spaces between/around the X's
_ _ X _ X _ _ X _ _
This creates 4 groups with a total of 7 elements.
$(2, 1, 2, 2)$

Formally

Let B be the set of $(t-1)$-element subsets of $\{1, 2, \dots, n+t-1\}$
Let A be the set of t-type multisets of size n.

$f: B \to A$
Input $S = \{s_1 < s_2 < \cdots < s_{t-1}\}$
Let $m_1 = s_1 - 1, m_i = s_i - s_{i-1} - 1$ $for\ 2 \leq i \leq t-1$
$m_t = n + t - 1 - s_{t-1}$
Output $(m_1, m_2, \dots, m_t)$

$g: A \to B$
Input $(m_1, m_2, \dots, m_t) \in A$
For $1 \leq i \leq t-1$ $let\ s_i = m_1 + m_2 + \cdots + m_i + i$
Output $\{s_1, s_2, \dots, s_{t-1}\}$

Check
* for all $\mu \in A: f(g(\mu)) = \mu$
* for all $S \in B: g(f(S)) = S$

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Back to General Problem

We've seen that for all $t \geq 1$

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Coefficient is a polynomial in n of degree $t - 1$

Example

$$\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}+\frac{C}{(1-\beta x)^2}+\frac{D}{(1-\beta x)^3}$$

$$= A\sum_{(n=0)}^{\infty}\alpha^n x^n + B\sum_{n=0}^{\infty}\beta^n x^n + C\sum_{(n=0)}^{\infty}\binom{n+1}{1}\beta^n x^n + D\sum_{n=0}^{\infty}\binom{n+2}{2}\beta^n x^n$$

$$= \sum_{n=0}^{\infty}(Aa^n + (Bc_0 + Cc_1 + Dc_2)\beta^n)x^n$$

$c_i = \binom{n+i}{i}$ is a polynomial of degree $\leq i$

Binary Strings

September-19-11 1:30 PM

Binary Strings

$\{0,1\}^*$ is the set of all finite strings of 0s and 1s
$\sigma = b_1 b_2 \dots b_n$ with each $b_i \in \{0,1\}$ is a word

$\mathcal{L} \subseteq \{0,1\}^*$ is a language

Length

The length of a word $\sigma \in \{0,1\}^*$ is the number of letters in it, $l(\sigma)$

Language Generating Function

Generating Function of a language $\mathcal{L}$ is

$$L(x) = \sum_{\sigma \in \mathcal{L}} x^{l(\sigma)} = \sum_{n=0}^{\infty} \left(\sum_{\substack{\sigma \in \mathbb{L} \\ l(\sigma)=n}} 1 \right) x^n$$

For every $n \in \mathbb{N}$: the coefficient of x^n in $L(x)$ is the number of words in $\mathcal{L}$ of length n.

Constructing Languages

Union

$A \cup B = \{\sigma \in \{0,1\}^* : \sigma \in A \text{ or } \sigma \in B\}$

Concatenations

$AB = \{\alpha\beta : \alpha \in A \text{ and } \beta \in B\}$
is the concatenation of A and B

Unambiguous Concatenation

The concatenation AB is unambiguous if each word AB is constructed exactly once in the form $\sigma = \alpha\beta$ with $\alpha \in A, \beta \in B$.
That is, AB is in bijection with $A \times B$

Iteration

If A is a language then A* is the iteration of A, consisting of all words $\sigma = \alpha_1 \alpha_2 \dots \alpha_k$ for some $k \in \mathbb{N}$, with $\alpha_i \in A$ for each $1 \le i \le k$

Ex: $\{0,1\}^*$ is an instance of iteration

Unambiguous Iteration

A^* is unambiguous if every word $\sigma \in A^*$ can be written as $\sigma = \alpha_1 \alpha_2 \dots \alpha_k$ for a unique value of $k \in \mathbb{N}$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in A$.

Sum Lemma

If $A, B \subseteq \{0,1\}^*$ and $A \cap B = \emptyset$ then the generating function for $A \cup B = A(x) + B(x)$

Product Lemma

For $A, B \subseteq \{0,1\}^*$, if AB is unambiguous then the the generating function for AB is $A(x)B(x)$

Iteration lemma

If $A \subseteq \{0{,}1\}^*$ and A^* is unambiguous, then the generating function for A^* is $\frac{1}{1-A(x)}$.

A game

- Player wagers n dollars
- Player flips a fair coin n times
- If Player hits a run of 3 (or more) heads, he wins \$10
- Otherwise he loses the wager (\$n)

1st question: What is the smallest value of n for which this is profitable for Player?
2nd question: Suppose House pays the player $w(n)$ dollars when Player hits HHH. What function $w(n)$ makes the game completely fair?

Example, n=3

Expected profit of Player is

$$\frac{7 \times (-3) + 1 \times (10)}{8} = -\frac{11}{8}$$

n=4

2^4 outcomes
3 outcomes have ≥ 3 heads
Expected profit

$$\frac{13 \times (-4) + 3 \times (10)}{16} = -\frac{22}{16} = -\frac{11}{8}$$

Let g_n be the number of binary strings of length n which do not contain 000 as a substring.
$G \subseteq \{0,1\}^*$ is the set of all binary strings that don't contain 000 as a substring.

Proof of Sum Lemma

$$\sum_{\sigma \in A \cup B} x^{l(\sigma)} = \sum_{\sigma \in A} x^{l(\sigma)} + \sum_{\sigma \in B} x^{l(\sigma)} = A(x) + B(x)$$

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Proof of Product Lemma

$$\sum_{\sigma \in AB} x^{l(\sigma)} = \sum_{\alpha \in A} \sum_{\beta \in B} x^{l(\alpha)+l(\beta)} = \left(\sum_{\alpha \in A} x^{l(\alpha)} \right) \left(\sum_{\beta \in B} x^{l(\beta)} \right) = A(x)B(x)$$

Proof of Iteration Lemma

Generating function for A^* is

$$\sum_{\sigma \in A^*} x^{l(\sigma)} = \sum_{k=0}^{\infty} \sum_{\alpha_1, \alpha_2, \dots, \alpha_k \in A^k} x^{l(\alpha_1 \alpha_2 \dots \alpha_k)} = \sum_{(k=0)}^{\infty} \sum_{\alpha_1 \in A} \sum_{\alpha_2 \in A} \dots \sum_{\alpha_k \in A} x^{l(\alpha_1)+l(\alpha_2)+\dots+l(\alpha_k)}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{\alpha \in A} x^{l(\alpha)} \right)^k = \sum_{k=0}^{\infty} A(x)^k = \frac{1}{1-A(k)}$$

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Language Expressions

September-21-11 1:32 PM

Rational Languages

- $\emptyset, \{0\}, \{1\}$ are rational languages.
- If A, B are rational then so are $A \cup B, AB, A^*$

Regular Expression

Any expression involving $\{0\}, \{1\}, \emptyset, \cup, \cdot, ^*$ that is well-formed.
Every regular expression determines a rational language.

Unambiguous

Every string can be constructed in exactly one way

Theorem

Every rational language has an unambiguous regular expression.

Proof: Take a graduate CS course

Notation

$(0 \cup 1)^*$ instead of $(\{0\} \cup \{1\})^*$
$\epsilon = ()$ —string of length 0
$\emptyset = \{\}$ - null set

Block

A block in a binary string $\sigma = b_1 b_2 \dots b_n$ is a substring of consecutive equal letters that is maximal w.r.t length.

Note:

Maximal, not maximum
Blocks are always non-empty

Block Decompositions

0*(1*10*0)*1* and 1*(0*01*1)*0* are block decompositions for the set of all binary strings. **Block decompositions always unambiguous**.

Examples of regular expressions

$\{0,1\}^* = (\{0\} \cup \{1\})^*$ is an unambiguous regular expression.
The generating function of $\{0\} \cup \{1\}$ is $2x^1$
By iteration:

$\{0,1\}^*$ has generating function $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$

0^*0 is $\{0\}^*\{0\} = \{0, 00, 000, 0000, \dots\}$
has generating function

$$= \frac{x}{1-x} = \frac{1}{1-x} \times x$$

Blocks

Want to split a binary string into blocks. Can have a block of 1s followed by a block of 0s, all repeated.

Regular expression:
block of 0s: 0*0
block of 1s: 1*1
Block of 1s followed by block of 0s: (1*1)(0*0)

Therefore, the regular expression (1*10*0)* allows constructing of any string that does not start with 0 or end with 1

Claim: 0*(1*10*0)*1* produces all strings unambiguously
Generating function:

$$0^*, 1^* \to \frac{1}{1-x}$$

$$0^*01^*1 \to \left(\frac{x}{1-x}\right)^2$$

$$0^*(1^*10^*0)1^* = \frac{1}{1-x} \cdot \frac{1}{1-\left(\frac{x}{1-x}\right)^2} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^2 - x^2} = \frac{1}{1-2x}$$

Coin Flipping Game

Let $G \subseteq \{0,1\}^*$ be the set of binary strings that don't contain 000 as a substring.
$(\epsilon \cup 0 \cup 00)\left(1^*1(0 \cup 00)\right)^* 1^*$
A block decomposition for G
Generating function:

$$(1+x+x^2) \cdot \frac{1}{1-\left(\frac{1}{1-x} \cdot (x+x^2)\right)} \cdot \frac{1}{1-x} = \frac{1+x+x^2}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} g_n x^n$$

Now use partial fractions to get a formula for g_n
$g_0 = 1$
$g_1 - g_0 = 1 \Rightarrow g_1 = 2$
$g_2 - g_1 - g_0 = 1 \Rightarrow g_2 = 4$
$g_n = g_{n-1} + g_{n-2} + g_{n-3}$

Fair Game

- Player wages \$n to flip n coins
- If no HHH, then player loses \$n
- If there is some HHH player wins R_n dollars

Chose R_n so that the game is fair - expected value is 0

$G \subseteq \{H, T\}^*$, strings that do not contain HHH
g_n: number of strings of length n in G
Block decomposition:
$T^*\left((H \cup HH)T^*T\right)^*(\varepsilon \cup H \cup HH)$

$$G(x) = \sum_{n=0}^{\infty} g_n x^n = \frac{1+x+x^2}{1-x-x^2-x^3}$$

Expected value of coin-flipping game, wagering \$n

$$0 = \frac{1}{2^n}\left((2^n - g_n)R_n + g_n(-n)\right)$$

$$n g_n = (2^n - g_n)R_n$$

$$R_n = \frac{n g_n}{2^n - g_n}$$

$1 - x - x^2 - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x)$
$\alpha, \beta \approx -0.4196 \pm 0.6063i$
$\gamma \approx 1.839$
By partial fractions
$g_n = A\alpha^n + B\beta^n + C\gamma^n, for\ constants\ A, B, C$
Since $|\alpha|, |\beta| < |\gamma| < 2$
$\frac{g_n}{2^n} \to 0\ as\ n \to \infty$

$R_n = n\frac{g_n}{2_n}\left(\frac{1}{1-\frac{g_n}{2_n}}\right) \to 0\ as\ n \to \infty$

Since $\frac{ng_n}{2^n} \to 0\ as\ n \to \infty$ l'Hopital's Rule

Fair reward for n coin flips is

$R_n = \frac{ng_n}{2^n - g_n} \to 0$

2-Variable Generating Function

September-23-11 1:35 PM

Example

What is the expected number of blocks among all binary strings of length n?

For each string, two pieces of information: the length $l(\sigma)$ and the # of blocks $b(\sigma)$

Use Two-Variable generating function

$$B(x,y) = \sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)} y^{b(\sigma)}$$

Block decomposition of {0,1}*: 0*(1*10*0)1*

0*0 and 1*1 produce blocks of 0s or 1s respectively

$0^* = \varepsilon \cup 0^*0$

$1^* = \varepsilon \cup 1^*1$

Blocks of 0s 0*0 = {0, 00, 000, ...}

$\rightarrow (x + x^2 + x^3 + \cdots)y = \frac{xy}{1-x}$

Blocks of 1s 1*1 = {1, 11, 111, ...}

$\rightarrow \frac{xy}{1-x}$ similarly

$$0^* \rightarrow x^0 y^0 + \frac{xy}{1-x} = 1 + \frac{xy}{1-x} = \frac{1 + x(y-1)}{1-x}$$

$1^* \rightarrow same$

From the block decomposition,

$$B(x,y) = \left(1 + \frac{xy}{1-x}\right)^2 \left(\frac{1}{1 - \left(\frac{xy}{1-x}\right)^2}\right) = \frac{(1-x+xy)^2}{(1-x)^2 - (xy)^2} = \frac{1-x+xy}{1-x-xy}$$

$$B(x,1) = \sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)}\, 1^{b(\sigma)} = \sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)} = \frac{1}{1-2x}$$

$$\frac{\delta}{\delta y} B(x,y)\Big|_{y=1} = \sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)} b(\sigma) y^{b(\sigma)-1}\Big|_{y=1} = \sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)} b(\sigma) = \sum_{n=0}^{\infty} \left(\sum_{\substack{\sigma \in \{0,1\}^* \\ l(\sigma)=n}} b(\sigma) \right) x^n$$

For every $n \in \mathbb{N}$, the total number of blocks among all binary string of length n is

$$[x^n] \frac{\delta}{\delta y} B(x,y)\ \Big|_{y=1}$$

$$\frac{\delta}{\delta y}\left(\frac{1-x+xy}{1-x-xy}\right)\Big|_{y=1} = \left(\frac{x}{1-x-xy} + \frac{(1-x+xy)(-1)(-x)}{(1-x-xy)^2}\right)\Big|_{y=1} = \frac{x(1-2x)+x}{(1-2x)^2} = \frac{2x-2x^2}{(1-2x)^2}$$

$$= \frac{2x}{(1-2x)^2} - \frac{2x^2}{(1-2x)^2}$$

$$= 2\sum_{n=0}^{\infty} \binom{n+1}{1} 2^n x^{n+1} - 2\sum_{n=0}^{\infty} \binom{n+1}{1} 2^n x^{n+2} = 0x^0 + 2x^1 = \sum_{k=2}^{\infty} (k2^k - (k-1)2^{k-1})$$

So for n≥ 2 the total # of blocks among all binary strings of length n is $n2^n - (n-1)2^{n-1} = (n+1)2^{n-1}$

So the average # of blocks per binary string of length n is

$$\frac{(n+1)2^{n-1}}{2^n} = \frac{n+1}{2}$$

Alternate Method

Number of blocks, for string of length n

$b_1 b_2 b_3 \ldots b_n$

First bit gives 2 possible blocks, every successive bit either is the same block or ads another block.

$$\sum_{\sigma \in \{0,1\}^n} x^{b(\sigma)} = 2x(1+x)(1+x)\ldots(1+x) = 2x(1+x)^{n-1}$$

$$\frac{d}{dx} 2x(1+x)^{n-1}\Big|_{x=1} = 2(1+x)^{n-1}\Big|_{x=1} + 2x(n-1)(x+1)^{n-2}\Big|_{x=1} = 2^n + 2^{n-1} = (n+1)2^{n-1}$$

So average $b(\sigma)$ among all 2^n $\sigma \in \{0,1\}^n$ is $\frac{n+1}{2}$

Similarly, for strings $\sigma \in \{1,2,\ldots,k\}^n$

$$\sum_{\sigma \in \{1,2,\ldots,k\}} x^{b(\sigma)} = kx(1+(k-1)x)^{n-1}$$

Average # of blocks among all $\sigma = \{1,2,\ldots,k\}^n$ is

$$\frac{1}{k^n}\frac{d}{dx} kx(1+(k-1)x)^{n-1}\Big|_{x=1}$$

Context-Free Grammars

September-26-11 1:32 PM

Proposition

If $\mathcal{L} \subseteq \{0,1\}^*$ is a rational language, then

$$L(x) = \sum_{\sigma \in \mathcal{L}} x^{l(\sigma)}$$

is a rational function (quotient of two polynomials).

Context Free Grammars

Initial symbol I
Production rules

Binomial Series Expansion

For an $\alpha \in \mathbb{C}$

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

Where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$

Proof

Taylor series expansion of $(1+x)^\alpha$. Coefficient of x^n *is*

$$\frac{1}{n!}\frac{d^n}{dx^n}(1+x)^\alpha\Big|_{x=0} = \frac{1}{n!}\alpha(\alpha-1)\dots(\alpha-n+1) = \binom{\alpha}{n}$$

Proofoid of Proposition

$\mathcal{L} = A \cup B \ or \ \mathcal{L} = AB \ or \ \mathcal{L} = A^*$

By induction, $A(x), B(x)$ are rational functions. Each operation takes rational functions to rational functions, so $\mathcal{L}(x)$ is rational too. ■

Converse is false

$M = \{\varepsilon, 01, 0011, 000111, \dots\} = \{0^k 1^k : k \in \mathbb{N}\}$

M is a set of binary strings with generating function $M(x) = \frac{1}{1-x^2}$ a rational function.

But M is not a rational language.

Context Free Grammar Example

Initial symbol I
Production rule $I \to \epsilon \cup 0I1$
Terminal symbols 0,1
Replace I by either ϵ or 0I1

Keep doing that until only terminal symbols remain
I → 0I1 → 00I11 → 000I111 →
ϵ 01 0011 000111

Let $\mathcal{D} \in \{0,1\}^*$ be generated by the CFG:
$I \to \epsilon \cup 0I1I$
ϵ, 01, 0011, 0101, 010011, 000111, 001101, …
Equivalently replace 0 by (and 1 by)
$I \to \epsilon \cup$ (I)I
This generates all well-formed parenthesizations.

Let $D(x) = \sum_{\sigma \in \mathcal{D}} x^{l(\sigma)}$

The CFG I→ ϵ ∪0I1I implies that
0→ x, I→ $D(x)$ 1→ x, I→ $D(x)$

$D(x) = 1 + x^2 (D(x))^2$

$D = 1 + x^2 D^2$

$0 = x^2 D^2 - D + 1$

$$D = \frac{1 \pm \sqrt{1-4x^2}}{2x^2}$$

How to expand $\sqrt{1-4x^2}$ as a power series in x?

$$\sqrt{1-4x^2} = (1-4x^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^{2n}$$

$n = 0$: $\binom{\frac{1}{2}}{0}(-4)^0 = 1$

$n \geq 1$:

$$\binom{\frac{1}{2}}{n}(-4)^n = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-n+1\right)}{n!}(-1)^n 2^n 2^n$$

$$= \frac{(1)(-1)(-3)(-5)\dots(-2n+3)}{n!}(-1)^n 2^n = -\frac{1 \times 3 \times 5 \times \dots \times (2n-3)}{n!} 2^n \times \frac{n!}{n!}$$

$$= -\frac{\left(1 \times 3 \times 5 \times \dots \times (2n-3)\right) \times \left(2 \times 4 \times 6 \times \dots \times (2n)\right)}{n!\,n!} = \frac{(-2n)(2n-2)!}{n!\,n!} = -\frac{2}{n}\binom{2n-2}{n-1}$$

In summary

$$\sqrt{1-4x^2} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^{2n}$$

Take -ve sign in $D(x)$ to get nonnegative results

$$D(x) = \frac{1}{2x^2}\left(1 - \left(1 - 2\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{2n}\right)\right) = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{2n-2} = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{2n}$$

Thus for all $n \in \mathbb{N}$ the number of well-formed parenthesizations with n '(' and n')' is $\frac{1}{n+1}\binom{2n}{n}$

Paths

September-28-11 1:30 PM

Binomial Series

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

for any $\alpha \in \mathbb{C}$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$$

Special Cases

1.
$\alpha = d$ a positive integer
$\binom{d}{n} = 0$ if n > d

So $(1+x)^d = \sum_{n=0}^{d} \binom{d}{n} x^n$

2.
$\alpha = -t$ a negative integer

$$\frac{1}{(1-x)^t} = \sum_{m=0}^{\infty} \binom{m+t-1}{t-1} x^m$$

Check that (exercise)

$$(-1)^m \binom{-t}{m} = \binom{m+t-1}{t-1}$$

Catalan Numbers

$$\frac{1}{n+1}\binom{2n}{n}$$

Lattice Path

A path on the grid which can only move N or E.

There are $\binom{a+b}{b} = \binom{a+b}{a}$ lattice paths from (0, 0) to (a,b)

Dyck Path

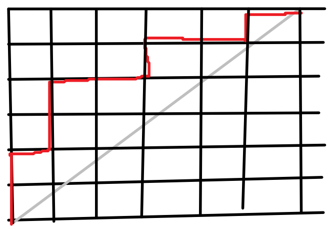
A lattice path which always stays above the $x = y$ line.
There are $\frac{1}{n+1}\binom{2n}{n}$ Dyck paths from (0, 0) to (n, n)

Catalan Numbers

$$\frac{1}{n+1}\binom{2n}{n}$$

is the formula for the Catalan numbers. e.g. the number of well-formed parenthesizations.
(()(())())()
Interpret as a lattice path
$(\rightarrow N : (x,y) \rightarrow (x, y+1)$
$) \rightarrow E : (x,y) \rightarrow (x+1, y)$
Start at $(0,0)$ and end at (n,n)

So the set of all well-formed parenthesizations is equivalent to the number of lattice paths from (0, 0) to (n, n) that stays above the $x = y$ line.
This is a Dyck Path.



Second Proof of # of Dyck Paths

Consider $\mathcal{L}(n,n)$ the set of all lattice paths from (0, 0) to (n, n)
Let $\mathcal{D}_n$ be the Dyck paths from (0, 0) to (n, n)
let $\mathcal{G}_n$ be the others.

So $\mathcal{L}(n,n) = \mathcal{D}_n \cup \mathcal{G}_n$ is a disjoint union
$|\mathcal{L}(n,n)| = \binom{2n}{n}$
We need only count $|\mathcal{G}_n|$ and subtract.
Consider any lattice path
$P: s_1 s_2 \dots s_{2n}$ in $\mathcal{G}_n$

Since $P \notin \mathcal{D}_n$ there is a first E step at which P goes below the diagonal $x = y$. Call it s_b for some $1 \le b \le 2n$

Construct the path
$P^*: t_1 t_2 \dots t_{2n}$

$$t_i = \begin{cases} s_i \; if \; 1 \le i \le b \\ N \; if \; s_i = E \; and \; b+1 \le 1 \le 2n \\ E \; if \; s_i = N \; and \; b+1 \le 1 \le 2n \end{cases}$$

Claim: P^* is a lattice path from (0, 0) to (n+1, n-1)

Conversely, every lattice path $Q: p_1 p_2 \dots p_{2n}$ from (0, 0) to (n+1, n-1) has a first E step p_j that goes below the diagonal x=y. Reverse the procedure $Q \rightarrow Q^*$ Result Q^* is in $\mathcal{G}_n$ (exercise)

We have a bijection $\mathcal{G}_n \leftrightarrows \mathcal{L}(n+1, n-1)$ hence $|\mathcal{G}_n| = |\mathcal{L}(n+1, n-1)| = \binom{2n}{n-1}$
Hence finally

$$|\mathcal{D}_n| = \binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n+1)!\,(n-1)!} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}$$

Analogously, lattice paths from (0, 0) to (a, b) where $0 \le a \le b$ that stay on or above the line x=y
How many such paths are there?
There are $\binom{a+b}{b}$ lattice paths from (0, 0) to (a, b)
Consider such a lattice path P that does go below the line $x = y$. $P: s_1 s_2, \dots, s_{a+b}$
Let s_i be the first step at which P goes below the diagonal
Let $\bar{N} = E$ and $\bar{E} = N$ and $p^*: s_1 \dots s_i \overline{s_{i+1}}, \overline{s_{i+2}} \dots \overline{s_{a+b}}$
p* ends at (b+1, a-1), strictly below $x = y$ since $a \le b$
This is a bijection between bad lattice paths to (a, b) and all lattice paths to (b+1, a-1)

Hence the number of good lattice paths to (a, b) is $\binom{a+b}{b} - \binom{a+b}{b+1}$
Where $a = b$ equal formula for dyck path

Ternary Strings

September-30-11 1:47 PM

Example

Enumerate strings in {a, b, c}* that don't contain aa as a substring

Look at block decomposition for binary string
0*(1*10*0)*1*
Interpret 0 as a, 1 as $b \cup c$

$a^*\left((b \cup c)^*(b \cup c)a^*a\right)^*(b \cup c)^*$
Is a regular expression for $\{a, b, c\}^*$ that produces as block by block.
Just need to modify this to avoid substring aa
$(\epsilon \cup a)\left((b \cup c)^*(b \cup c)a\right)^*(b \cup c)$

$$\sum_{\sigma \in S} x^{l(\sigma)} = (1+x)\left(\frac{1}{1-\left(\frac{1}{1-2x}\right)(2x)(x)}\right)\left(\frac{1}{1-2x}\right) = \frac{1+x}{1-2x-2x^2} \rightarrow partial\ fractions$$

or

$$c_n - 2c_{n-1} - 2c_{n-2} = \begin{cases} 1, & n = 0 \\ 1, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

$c_0 = 1$
$c_1 - 2c_0 = 1 \Rightarrow c_1 = 3$
$c_n = 2c_{n-1} + 2c_{n-2}$

n	0	1	2	3	4	5
c_n	1	3	8	22	60	164

Example

Enumerate strings in {a, b, c}* with no two consecutive equal letters, $\mathcal{D}$
Low tech solution
$c_0 = 1$
$c_1 = 3$
$c_n = 2c_{n-1} = 3 \times 2^{n-1}\ for\ n \geq 1$

$$\sum_{n=0}^{\infty} c_n x^n = 1 + 3\sum_{n=1}^{\infty} 2^{n-1}x^n = 1 + \frac{3x}{1-2x} = \frac{1+x}{1-2x}$$

More information

Keep track of #a, #b, #c in string
$m_a(\sigma) = $ # of a's in string σ
Similarly for m_b, m_c

$$D(x,y,z) = \sum_{\sigma \in \mathcal{D}} x^{m_a(\sigma)} y^{m_b(\sigma)} z^{m_c(\sigma)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{i,j,k} x^i y^j z^k$$

Consider any string $\sigma \in \{a, b, c\}$. "Squish" each block into a single letter.
E.g. $\sigma = bbcccaccbbbaaa \quad squish(\sigma) = BCACBA \in \mathcal{D}$

The set of words $\sigma \in$ {a, b, c}* that get squished onto $\alpha \in \mathcal{D}$ is obtained by regarding
A as a block of a's A=a*a, B=b*b, C=c*c
$(a \cup b \cup c)^*$ is a regular expression for $\{a, b, c\}^*$

$$\frac{1}{1-(x+y+z)} = \sum_{\sigma \in \{a,b,c\}^*} x^{m_a(\sigma)} y^{m_b(\sigma)} z^{m_c(\sigma)} = \sum_{\alpha \in \mathcal{D}}\left(\sum_{\sigma \in squish^{-1}(\alpha)} x^{m_a(\sigma)} y^{m_b(\sigma)} z^{m_c(\sigma)}\right)$$
$$= \sum_{\alpha \in \mathcal{D}}\left(\frac{x}{1-x}\right)^{m_A(\alpha)}\left(\frac{y}{1-y}\right)^{m_B(\alpha)}\left(\frac{z}{1-z}\right)^{m_C(\alpha)} = D\left(\frac{x}{1-x}, \frac{y}{1-y}, \frac{z}{1-z}\right)$$

Change variables

$$X = \frac{x}{1-x}, Y = \frac{y}{1-y}, Z = \frac{z}{1-z}$$

$$X - xX = x \Rightarrow X = x + xX = x(1+X) \Rightarrow x = \frac{X}{1+X}$$

$$D(X,Y,Z) = \frac{1}{1-\left(\frac{X}{1+X} + \frac{Y}{1+Y} + \frac{Z}{1+Z}\right)}$$

A quotient of polynomials in X,Y,Z

More generally for strings $\mathcal{D} \subseteq \{1, 2, \dots, b\}^*$ with no two consecutive equal letters

$$\frac{1}{1-(x_1+x_2+\cdots+x_b)} = D\left(\frac{x_1}{1-x_1}, \frac{x_2}{1-x_2}, \dots, \frac{x_b}{1-x_b}\right)$$

$$D(x_1, x_2, \dots, x_b) = \left[1 - \sum_{i=1}^{b} \frac{x_i}{1+x_i}\right]^{-1}$$

n-ary Strings

October-03-11 1:33 PM

Example

Among all 2^n binary strings of length n, what is the average number of times that 011 occurs as a substring.

Block decomposition:
1*(0*01*1)0* is almost ideal, 1*(0*01u0*(011)1*)*0*

$l(\sigma)$ length of sigma, $r(\sigma)$ number of 011 in σ

$$G(x,y) = \sum_{\sigma \in \{0,1\}^*} x^{l(\sigma)} y^{r(\sigma)} = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-\left(\frac{x^2}{1-x}+\frac{x^3}{(1-x)^2}y\right)}\right)\left(\frac{1}{1-x}\right)$$

$$= \left((1-x)^2 - x^2(1-x) - x^3y\right)^{-1} = (1-2x+x^2-x^2+x^3-x^3y)^{-1} = \frac{1}{1-2x+x^3(1-y)}$$

Sum of $r(\sigma)$ over all $\sigma \in \{0,1\}^*$ in

$$[x^n]\frac{\delta}{\delta y}G(x,y)\Big|_{y=1} = \frac{(-1)(-x^3)}{(1-2x)^3} = \frac{x^3}{(1-2x)^2} = x^3\sum_{n=0}^{\infty}\binom{n+1}{1}2^n x^n = \sum_{n=0}^{\infty}(n+1)2^n x^{n+3}$$

$$= \sum_{n=3}^{\infty}(n-2)2^{n-3}x^n$$

Average # of occurrences of 011 among all $\sigma \in \{0,1\}^n$ is

$$\begin{cases} \frac{(n-2)2^{n-3}}{2^n} = \frac{n-2}{8}, & n \geq 3 \\ 0, & 0 \leq n \leq 2 \end{cases}$$

Block Patterns for b-ary strings

$\mathcal{D} \subseteq \{1,2,\dots,b\}^*$ strings with no two consecutive equal letters.
$x_1, x_2, \dots, x_b$ variables
$m_i(\sigma)$ is the # of times letter i occurs in σ
Notation: $x^\sigma = x_1^{m_1(\sigma)} x_2^{m_2(\sigma)} \dots x_b^{m_b(\sigma)}$

$$D(x_1,\dots,x_b) = \sum_{\sigma\in\mathcal{D}} x^\sigma = \left(1-\sum_{i=1}^{b}\frac{x_i}{1+x_i}\right)^{-1}$$

Proof:

squish: $\{1,\dots,b\}^* \to \mathcal{D}$ by replacing each block of i's by a single i
For $\alpha \in \mathcal{D}$, the $\sigma \in \{1,2,\dots,b\}^*$ that gets squished to α are obtained from α by replacing i by i^*i for all $1 \leq i \leq b$ generating function for i^*i is $\frac{x_i}{1-x_i}$

So

$$\frac{1}{1-(x_1+x_2+\dots+x_b)} = D\left(\frac{x_1}{1-x_1},\frac{x_2}{1-x_2},\dots,\frac{x_b}{1-x_b}\right)$$

Invert the variables $y_i = \frac{x_i}{1-x_i}$ iff $x_i = \frac{y_i}{1+y_i}$

So $D(y_1,y_2,\dots,y_b) = \left(1-\sum_{i=1}^{b}\frac{y_i}{1+y_i}\right)^{-1}$

Strings in $\mathcal{D}$ are block patterns. x_i in $\mathcal{D}$ marks either

- A single i in $\alpha \in \mathcal{D}$
- A block of $i's$ in $\sigma \in \{1,2,\dots,b\}^*$

Example

What is the generating function for S, strings $\sigma \in \{1,2,3\}^*$ such that

- Blocks of 1s have odd length
- Blocks of 2s have length ≤ 2
- Blocks of 3s have length ≥ 2

$D(y_1,y_2,y_3)$ where y_1 marks a block of is
(11)*1 $\Rightarrow y_1 = \frac{x_1}{1-x_1^2}$
(2u22) $\Rightarrow y_2 = x_2 + x_2^2$
3*33 $\Rightarrow y_3 = \frac{x_3^2}{1-x_3}$

$$S(x_1,x_2,x_3) = D(y_1,y_2,y_3) = \left(1-\frac{x_1}{1-x_1^2}-(x_2+x_2^2)-\frac{x_3^2}{1-x_3}\right)^{-1} = \sum_{\sigma\in S} x^\sigma$$

If we only want the length of each $\sigma \in S$ e.g. $x_1 = x_2 = x_3 = t$

$$S(t,t,t) = \sum_{\sigma\in S} t^{l(\sigma)} = \left(1-\frac{t}{(1-t)^2}-t(1+t)-\frac{t^2}{1-t}\right) = \frac{1-t^2}{1-2t-3t^2+t^4}$$

$$s_n - 2s_{n-1} - 3s_{n-2} + s_{n-4} = \begin{cases} 1, & n=0 \\ 0, & n=1 \\ -1, & n=2 \\ 0, & n\geq 3 \end{cases}$$

Keep going and get a recurrence relation.

Example

a_n crossings n steps from home on a rectangular grid (n is minimum distance)
$a_0 = 1$
$a_1 = 4$
$a_2 = 8$
$$a_n = \begin{cases} 1, & n = 0 \\ 4n, & n \geq 1 \end{cases}$$
$$\sum_{n=0}^{\infty} a_n x^n = 1 + 4\frac{x}{(1-x)^2}$$

a_n crossings n steps from home on a triangular grid (n is minimum distance)
$a_0 = 1$
$a_1 = 6$
$a_2 = 12$
$$a_n = \begin{cases} 1, & n = 0 \\ 6n, & n \geq 1 \end{cases}$$
$$\sum_{n=0}^{\infty} a_n x^n = 1 + 6\frac{x}{(1-x)^2}$$

Tile the plan with squares, 5 at a point.

Tessellations

October-05-11 2:03 PM

Regular Tessellations of the Plane

Let $k \geq 3$ and $d \geq 3$. Divide the plane into non-overlapping k-gons such that they meet along edges. At each corner d edges meet.

Question

Fix a "home vertex" v_0 in the $k = 4, d = 5$ regular tessellation of the (hyperbolic) plane. many vertices are at distance exactly n from v_0? Call it a_n

n	0	1	2	3	4
a_n	1	5	15		

At distance 2 there are 2 kinds of vertices.

- Some have 1 neighbour at distance 1
- Some have 2 neighbours at distance 1

Showed geometrically can't have ≥ 3 neighbours closer to base

Let b_n be the number of vertices at distance n from the base, with 1 earlier neighbour
Let c_n be the number of vertices at distance n from the base, with 2 earlier neighbours
For $n \geq 1, a_n = b_n + c_n$

$$n \geq 1: \begin{cases} b_{n+1} = 2b_n + c_n \\ c_{n+1} = a_n = b_n + c_n \end{cases}$$

$a_0 = 1$
$b_1 = 5, c_1 = 0$

$$\text{Let } A(x) = \sum_{n=0}^{\infty} a_n x^n, B(x) = \sum_{n=1}^{\infty} b_n x^n, C(x) = \sum_{n=1}^{\infty} c_n x^n$$

$$A(x) = 1 + \sum_{n=1}^{\infty} (b_n + c_n)x^n = 1 + B(x) + C(x)$$

$$B(x) = \sum_{n=1}^{\infty} b_n x^n = 5x + \sum_{n=2}^{\infty} (2b_{n-1} + c_{n-1})x^n = 5x + x\sum_{j=1}^{\infty} (2b_j + c_j)x^j$$

$$= 5x + x\big(2B(x) + C(x)\big)$$

$$C(x) = \sum_{n=1}^{\infty} c_n x^n = x\big(B(x) + C(x)\big)$$

$A = 1 + B + C$
$B = 5x + 2xB + xC$
$C = xB + xC$
Solve...

$$C = \frac{5x^2}{1 - 3x + x^2}$$

$$B = \frac{5x - 5x^2}{1 - 3x + x^2}$$

$$A = \frac{1 + 2x + x^2}{1 - 3x + x^2} = 1 + \frac{5x}{1 - 3x + x^2}$$

$$1 - 3x + x^2 = (1 - \alpha x)(1 - \beta x)$$

$$\alpha, \beta = \frac{3 \pm \sqrt{5}}{2}$$

$5x = A(1 - \beta x) + B(1 - \alpha x) = (A + B) - (A\beta + B\alpha)x$
$A + B = 0$
$A\beta + B\alpha = -5$

$$A(\beta - \alpha) = -5 \Rightarrow A = \frac{5}{\alpha - \beta}, B = -\frac{5}{\alpha - \beta}$$

$$\alpha - \beta = \frac{3 + \sqrt{5}}{2} - \frac{3 - \sqrt{5}}{2} = \sqrt{5}$$

$A = \sqrt{5}, B = -\sqrt{5}$

$$A(x) = 1 + \frac{\sqrt{5}}{1 - \alpha x} - \frac{\sqrt{5}}{1 - \beta x}$$

$$A(x) = 1 + \sqrt{5}\sum_{n=0}^{\infty}\left(\frac{(3 + \sqrt{5})}{2}\right)^n x^n - \sqrt{5}\sum_{n=0}^{\infty}\left(\frac{3 - \sqrt{5}}{2}\right)^n x^n$$

$$= 1 + \sum_{n=0}^{\infty}\left[\sqrt{5}\left(\frac{3 + \sqrt{5}}{2}\right)^n = \sqrt{5}\left(\frac{3 - \sqrt{5}}{2}\right)^n\right]x^n$$

So for $n \geq 1$ the number of vertices in the $k = 4, d = 5$ hyperbolic tessellation at distance n from the base is

$$a_n = \sqrt{5}\left(\frac{3 + \sqrt{5}}{2}\right)^n = \sqrt{5}\left(\frac{3 - \sqrt{5}}{2}\right)^n \Rightarrow \text{Integer closest to } \sqrt{5}\left(\frac{3 + \sqrt{5}}{2}\right)^n$$

Example

k=5, d=4
Four kinds of vertices in the k=5 d=4 case

- Base vertex
- One nbr closer to base, not on an equality (connects to same #) edge : p
- Two nbrs closer to base : q
- One nbr closer to base, is on an equality edge. : r

$$p(x) = \sum_{n=1}^{\infty} p_n x^n \ etc.$$

$p_1 = 4, q_1 = r_1 = 0$
$p_2 = 4, q_2 = 0, r_2 = 8$
$q_{n+2} = r_n$

More Tessellations

October-12-11 1:31 PM

Matrix Method

5 'types' of object O,A,B,C,D
and some succession rules.

Initial population: {O}

$0 \to 4A$

$A \to A, 2B$

$B \to B, C$

$C \to A, B, \frac{1}{2}D$

$D \to 2B$

$$P_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$P_n = M^n P_0$

k=5, d=4

Vertex Types

O: Origin

A: 1 neighbour closer to origin,

2 pentagons have apexes (unique vertex closest to origin) at this neighbour

B: 1 neighbour closer to origin, 1 neighbour at same distance

C: 1 neighbour closer to origin, that neighbour is of type B

D: 2 neighbours closer to origin

Descendants:

$0 \to \{4A\}$

$A \to \{A, 2B\}$

$B \to \{B, C\}$

$C \to \left\{A, B, \frac{1}{2}D\right\}$

$D \to \{2B\}$

$K(x) = \sum_{n=0}^{\infty} k_n x^n$ where there are k_n vertices of type k at distance n from the origin

$O(x) = 1$

For n$\geq$ 0

$a_{n+1} = 4o_n + a_n + c_n$

$A(x) = \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = \sum_{n=0}^{\infty} (4\,o_n + a_n + c_n)\,x^{n+1} = x[4\,O(x) + A(x) + C(x)]$

$b_{n+1} = 2a_n + b_n + c_n + 2d_n$

$B(x) = x[2A(x) + B(x) + C(x) + 2D(x)]$

$C(x) = x[B(x)]$

$D(x) = x\left[\frac{1}{2}C(x)\right]$

Solve:

$A = x(4 + A + C)$

$B = x(2A + B + C + 2D)$

$C = xB$

$D = \frac{1}{2}xC$

$A = 4x + xA + x^2B$

$B = 2xA + xB + x^2B + x^3B$

$(1-x)A = 4x + x^2B$

$2xA = (1 - x - x^2 - x^3)B$

$A = \frac{1 - x - x^2 - x^3}{2x}B$

$\frac{(1-x)(1 - x - x^2 - x^3)}{2x}B = 4x + x^2B$

$(1 - 2x + x^4)B = 8x^2 + 2x^3B$

$(1 - 2x - 2x^3 + x^4)B = 8x^2$

$B = \frac{8x^2}{1 - 2x - 2x^3 + x^4}$

$A = \frac{(1 - x - x^2 - x^3)4x}{1 - 2x - 2x^3 + x^4}$

$C = \frac{8x^3}{1 - 2x - 2x^3 + x^4}$

$D = \frac{4x^4}{1 - 2x - 2x^3 + x^4}$

$G(x) = 1 + A + B + C + D = \frac{1 + 2x + 4x^2 + 2x^3 + x^4}{1 - 2x - 2x^3 + x^4} = 1 + \frac{4(x + x^2 + x^3)}{1 - 2x - 2x^3 + x^4}$

Matrix Method

October-14-11 1:29 PM

Matrix Method

Find a set of types $\{1, 2, \ldots, t\}$

Succession Rules

For each type i, a weighted collection of successors:

$i \to \{c_1 1, c_2 2, \ldots, c_t t\}$

An object of type i gives rise to successors in the next generation:

c_i of type i

Initial Population

A column vector

$$p_0 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix}$$

a_i objects of type $i, (1 \le i \le t)$ in the initial population.

Goal

Determining the number of objects of type i in the n-th generation for all $(1 \le i \le t)$ and all $n \ge 0$

Construction

For each $n \in \mathbb{N}$ let p_n be the column vertex of length l with i-th entry equal to the # of type i objects in the n-th generation.

Let M be the $t \times t$ matrix such that $p_{n+1} = Mp_n \; \forall n \in \mathbb{N}$

The j-th column of M has i-th entry equal to the number of objects of type i occurring as successors to an object of type j

Since $p_{n+1} = Mp_n \; \forall n \in \mathbb{N}$

$P_n = M^n p_0$

Generating Function

Let $p(x) = \sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} M^n p_0 x^n = \left(\sum_{n=0}^{\infty} (xM)^n \right) p_0 = (I - xM)^{-1} p_0$

Reasoning

$S = 1 + A^2 + A^3 + \cdots$

$AS = A + A^2 + A^3 + \cdots$

$S - AS = 1$

$(1 - A)S = 1 \Rightarrow S = (1 - A)^{-1}$

Total Population

$$\vec{1}_t = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$Pop = \vec{1}_t p_n$

Generating function

$\vec{1}_t (T - xM)^{-1} p_0$

Note

$A^{-1} = \frac{1}{\det A} adj(A)$

$\det(I - xM) \neq 0$ so $I - xM$ is invertible since $I - xM$ is a polynomial in x and $\det(I - (1)M) = 1$

Example

$t = 3$ types $\{a, b, c\}$

Succession Rules $a \to \{a, b\}, b \to \{a, c\}, c \to \{a, a, a\}$

$$p_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$p_n = M^n p_0$

$$I - xM = \begin{bmatrix} 1 - x & -x & -3x \\ -x & 0 & 1 \\ 0 & -x & 1 \end{bmatrix}$$

$\det(I - xM) = 1 - x - x^2 - 3x^3$

$$adj(I - xM) = \begin{bmatrix} 1 & x + 3x^2 & 3x \\ x & 1 - x & 3x^2 \\ x^2 & x - x^2 & 1 - x - x^2 \end{bmatrix}$$

$P(x) = (I - xM)^{-1} p_0$

$$= \frac{1}{1 - x - x^2 - 3x^3} \begin{bmatrix} 1 & x + 3x^2 & 3x \\ x & 1 - x & 3x^2 \\ x^2 & x - x^2 & 1 - x - x^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{1 - x - x^2 - 3x^3} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

Total population generating function

$$\frac{1 + x + x^2}{1 - x - x^2 - 3x^3}$$

Total population w_n at generation n satisfies $w_n = 0 \; if \; n < 0$ and $w_n - w_{n-1} - w_{n-2} - 3w_{n-3} = \begin{cases} 1, & n = 0,1,2 \\ 0, & n \ge 3 \end{cases}$

$w_0 = 1$

$w_1 - w_0 = 1 \Rightarrow w_1 = 2$

$w_2 - w_1 - w_0 = 1 \Rightarrow w_2 = 4$

$w_n = w_{n-1} + w_{n-2} + 3w_{n-3}, n \ge 3$

Domino Tilings

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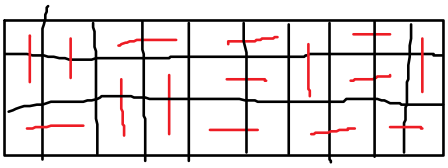
Domino Tilings

Count all ways of covering all squares of a $3 \times n$ rectangle with non-overlapping dominoes.

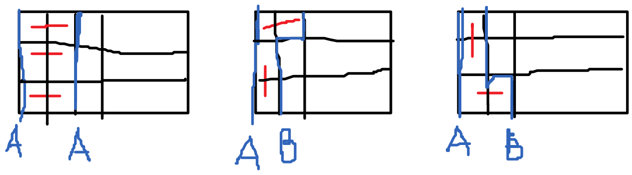


Columns instead of Dominoes

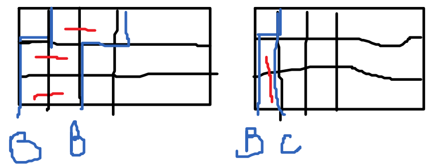
$A \to \{A_2, B_1\}$
$B \to \{A_1, B_2\}$
$Q = \begin{bmatrix} x^2 & x \\ x & x^2 \end{bmatrix}$

How

Consider all possible ways of covering the three leftmost squares:



Label the boundary types, but also keep track of the number of dominoes used in the subscript
$A \to \{A_3, B_2, B_2\}$



$B \to \{B_3, A_1\}$
Instead of xM we want a 2×2 matrix Q where Q_{ij} is the sum of x^k over all transitions from boundary j to boundary i using k dominoes.
$M = \begin{bmatrix} x^3 & x \\ 2x^2 & x^3 \end{bmatrix}$

Start with a 3xn domino tiling. Remove all dominoes that intersect the leftmost column (together with any dominoes they "force")

Repeat this to decompose each domino tiling uniquely as a sequence of "successions"
Two boundaries {A, B}
$A \to \{A_3, 2B_2\}$
$B \to \{A_1, B_3\}$
$M = \begin{bmatrix} x^3 & x \\ 2x^2 & x^3 \end{bmatrix}$
The (I, J) entry of M^n is the generating function from boundary J to boundary I using exactly n successions.
Sum over all $n \in \mathbb{N}$ since # of successions is arbitrary.

$$\sum_{n=0}^{\infty} M^n = (I - M)^{-1}$$

The generating function we want is $(I - M)^{-1}_{AA}$

$$\det(I - M) = \begin{vmatrix} 1 - x^3 & -x \\ -2x^2 & 1 - x^3 \end{vmatrix} = (1 - x^3)^2 - 2x^3 = 1 - 4x^3 + x^6$$

$adj(I - M)_{AA} = 1 - x^3$
Generating function for $3 \times n$ domino tilings is

$$G(x) = \sum_T x^{\#\, dominoes} = \frac{1 - x^3}{1 - 4x^3 + x^6}$$

$2 * \#dominoes = total\ \#\ squares = 3n$

$$n = \frac{2}{3}(\#\ dominoes), let\ x = t^{\frac{2}{3}}$$

$$G(x) = \sum_T t^{\frac{2}{3}\#\, dominoes} = \sum_{n=0}^{\infty} c_n t^n = \frac{1 - t^2}{1 - 4t^2 + t^4}$$

c_n domino tilings of a $3 \times n$ rectangle.

Examples

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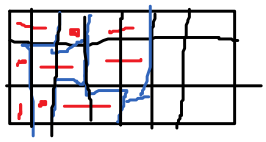
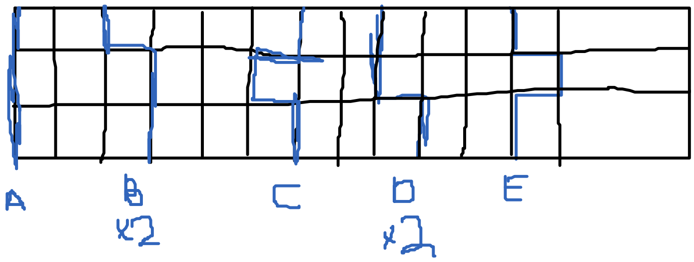
Example

Tilings of a 3xn rectangle using dominoes and 1x1 squares.



Possible boundary shapes



$J \to K_{a,b}$ Succession from boundary J to boundary K using a dominoes and b squares

$A \to \{A_{0,3}, A_{3,0}, 2D_{1,2}, E_{1,2}, 2B_{2,1}, C_{2,1}, 2A_{1,1}, 2D_{2,0}\}$
$B \to \{A_{0,1}, D_{1,0}\}$
$C \to \{A_{0,1}, E_{1,0}\}$
$D \to \{A_{0,2}, A_{1,0}, B_{2,0}, D_{1,1}, E_{1,1}\}$
$E \to \{A_{0,2}, C_{2,0}, 2D_{1,1}\}$

$$M = \begin{bmatrix} 2tu + t^3 + u^3 & u & u & t + u^2 & u^2 \\ 2t^2u & 0 & 0 & t^2 & 0 \\ t^2u & 0 & 0 & 0 & t^2 \\ 2t^2 + 2tu^2 & t & 0 & tu & 2tu \\ tu^2 & 0 & t & tu & 0 \end{bmatrix}$$

Example

$A \subseteq \{a, b, c\}^*$ Blocks of c's have odd length and does not contain aa or ab as a substring.
$a_n = \#$ of words of length n in A

Determine $\sum_{n=0}^{\infty} a_n x^n$

First determine the generating function for "block patterns" of A: the set of words in {a,b,c}* not containing any of aa, bb, cc, or ab.

$$P(x, y, z) = \sum_{\alpha \in P} x^{m_a(\alpha)} y^{m_b(\alpha)} z^{m_c(\alpha)}$$

Then replace each a in α with a block of a's, each b in α with a block of b's and each c in α by a block of c's. Keep track of the lengths of the blocks.
The lengths of the blocks are constrained:
no aa substring → block of a's is just a → t
block of b's → b*b → $\frac{t}{1-t}$
block of c's → (cc)*c → $\frac{t}{1-t^2}$

$$A(t) = \sum_{\sigma \in A} t^{l(\sigma)} = P\left(t, \frac{t}{1-t}, \frac{t}{1-t^2}\right)$$

Matrix Method

Find $P(x, y, z)$ using matrix method
$P \subseteq \{a, b, c\} *$ words not containing aa, bb, cc, or ab.
4 types: E,A,B,C: empty string, ends in a, ends in b, ends in c; respectively.
$E \to \{A, B, C\}$
$A \to \{C\}$
$B \to \{A, C\}$
$C \to \{A, B\}$
generate all the block patterns in A
M_{KL} is the sum over all transitions from K to L

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & x & x \\ y & 0 & 0 & y \\ z & z & z & 0 \end{bmatrix}$$

$$P(x, y, z) = [1 \quad 1 \quad 1 \quad 1](I - M)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{k=0}^{\infty} [1 \quad 1 \quad 1 \quad 1] M^k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{k=0}^{\infty} \sum_{\substack{\sigma \in P \\ l(\sigma) = k}} x^{m_a(\sigma)} y^{m_b(\sigma)} z^{m_c(\sigma)}$$

$$A(t) = P\left(t, \frac{t}{1-t}, \frac{t}{1-t^2}\right)$$

$$Q = I - M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & -t & -t \\ -\frac{t}{1-t} & 0 & 1 & -\frac{t}{1-t} \\ -\frac{t}{1-t^2} & -\frac{t}{1-t^2} & -\frac{t}{1-t^2} & 1 \end{bmatrix}$$

Example

Domino tiling. Start with A type boundary (straight line) and end with A type boundary.

Graph Theory

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Graph

A **graph** is a pair $G = (V, E)$ where V is a finite set, and E a set of 2-element subsets of V.
The elements of V are **vertices** and the elements of E are **edges**.

Isomorphism

An isomorphism φ from G to H is a function $\varphi: V(G) \to V(H)$ such that φ is a bijection (one-to-one and onto)

- φ is a bijection (one-to-one and onto)
- $\forall v, w \in V(G)$

$$\{v, w\} \in E(G) \Leftrightarrow \{\varphi(v), \varphi(w)\} \in E(H)$$

G and H are isomorphic, denoted by $G \cong H$, when there is an isomorphism φ from G to H.

Terminology

In a graph $G = (V, E)$
$v \in V$ is **incident** with $e \in E$ if $v \in e$
$v, w \in V$ are **adjacent** if $\{v, w\} \in E$
$e, f \in E$ are **adjacent** if $e \cap f = \{v\}$ for some $v \in V$
The **degree** of v is the number of edges incident with v.
Denoted $\deg_G(v)$
The **degree sequence** is the multiset $\{\deg_G(v) : v \in V\}$

Fact

If $\varphi: V(G) \to V(H)$ is an isomorphism then $\deg_H(\varphi(v)) = \deg_G(v)\ \ \forall v \in G$

Corollary

If $G \cong H$ then the degree sequences of G and H are the same.

Subgraph

$G = (V, E)$ is a graph
$J = (W, F)$ is a subgraph of G if $W \subseteq V, F \subseteq E$ and J is a graph.

K-Regular

A graph G is k-regular if every vertex has degree k.

Cycle

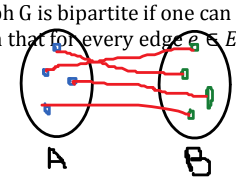
A cycle in G is a connected 2-regular subgraph.

Hamilton Cycle

A Hamilton cycle is a cycle through all the vertices.

Bipartite

A graph G is bipartite if one can write $V = A \cup B$ with $A \cap B = \emptyset$ such that for every edge $e \in E$ $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$



Equivalently, you can colour the graph with 2 colours such that every edge has one vertex of one colour and the other vertex having the other colour.

Proposition

a) If G is bipartite then every subgraph of G is bipartite.
b) Odd cycles are not bipartite

Corollary

If G contains an odd cycle, then G is not bipartite.

Notation

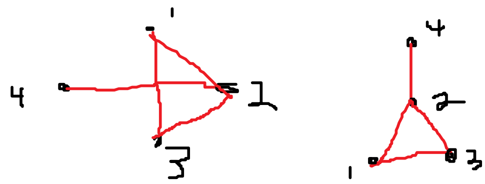
Complete graph: K_p

p vertices
$\binom{p}{2}$ edges; Every pair of vertices has an edges
$E = \{\{v_i, v_j\} : i \neq j\}$

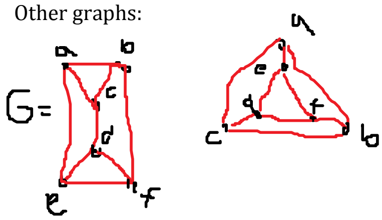
Complete bipartite graph: $K_{a,b}$

Graph Example

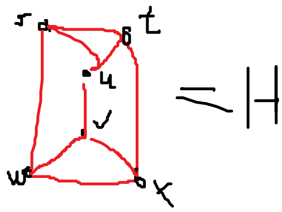
$G = (\{1,2,3,4\}, \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}\})$
Picture of G:



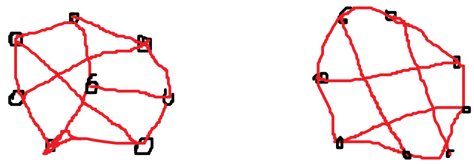
Other graphs:



These are the same graph: same vertices same edges. So the graphs are equal.



$G \neq H$ but they have the "same shape". i.e. they are isomorphic.



In this case G(left) contains an odd cycle while H(right) does not.
So $G \ncong H$

Proof of Proposition

(a) Let (A,B) be a bipartition for G and let $H = (W, F)$ be a subgraph of G. Then $(W \cap A, W \cap B)$ is a bipartition for H.

(b) Let C_n be an odd cycle with vertices $v_1, v_2, \dots, v_n$ (n odd) and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$

Suppose that (A,B) is a bipartition of C_n. Wlog we can assume $v_1 \in A$ (exchange A and B if necessary)
$\Rightarrow v_2 \in B \Rightarrow v_3 \in A \Rightarrow \cdots$
By induction from $1 \leq i \leq n$

$v_i \in A$ if i is odd
$v_i \in B$ if i is even

Since n is odd, $v_n \in A$. But then $\{v_n, v_1\} \subseteq A$ contradicting that (A,B) is a bipartition of G.
■

$a + b$ vertices
$A = \{v_1, \dots, v_a\}, B = \{w_1, \dots, w_b\}$
ab edges
$E = \left\{\{v_i, w_j\}: 1 \le j \le b, 1 \le i \le a\right\}$

Girth of G

if G has no cycles then $girth(G) = +\infty$
If G has cycles then $girth(G) = \min\{|E(C)|: C\ is\ a\ cyle\ in\ G\}$

Connectedness

October-24-11 1:32 PM

Walk

A walk in a graph is a sequence: $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$
Each $v_i \in V$, each $e_i \in E$ and $e_i = \{v_{i-1}, v_i\}$
Note that vertices and edges can be repeated.

Trail

A trail is a walk with no repeated edges

Path

A path is a walk with no repeated vertices.

Path ⇒ Trail, but Trail ⇏ Path

Closed & Cycle

A walk is closed if $v_0 = v_k$.
A cycle is (sometimes, incorrectly,) said to be a closed walk in which $v_0 = v_k$ is the only repeated vertex.

Reach

Define a relation R on the set V of vertices. vRw means there is a walk in G from v to w: $v = v_0 e_1 v_1 \dots e_k v_k = w$.
Say "v reaches w"

Fact

R is an equivalence relation.

Proof

Reflexive, Symmetric, Transitive

Connected Components

The equivalence classes of R on V induce subgraphs of G called the connected components of G

Induced Subgraph

For $S \subseteq V$, the subgraph of G induced by S has the vertex-set S and the edge set $F = \{e \in E : e \subseteq S\}$

Connected

The graph G is connected if it has exactly one connected component.

For graphs with at least one vertex, this is equivalent to:
$\forall v, w \in V$ there is a path from v to w (vRw)

Length of a Walk

The length of a walk is the number of edges in the walk.

Lemma

If there is a walk from v to w then there is a path from v to w.

Deleting an Edge

Deleting an edge from $G = (V, E)$ gives the graph $G\backslash$e $= (V, E\{e\})$

Minimally Connected Graph

A graph is minimally connected if it is connected but $G\backslash$e is not connected $\forall e \in E$.

Let $c(G)$ be the number of connected components of G. $e \in E$ is a **cut-edge** if $c(G\backslash\text{e}) > c(G)$

G is minimally connected if $c(G) = 1$ and every edge is a cut-edge.

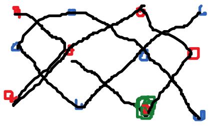
Lemma

Let $G = (V, E)$ be a graph. Let $e = \{x, y\} \in E$. Then e is a cut-edge of G iff e is not contained in a cycle of G.

Corollary

G is a minimally connected graph iff G is connected and contains no cycles.

Reach example



The green vertex can reach only the red vertices.

Proof of Lemma 1

Let W: $v = v_0 e_1 v_2 e_2 \dots e_k v_k = w$ be a walk from v to w which has a s few edges as possible.

If W has a repeated vertex $v_i = v_j$ with $0 \le i < j \le k$
Then W': $v_0 e_1 v_1 \dots e_i v_i e_{j+1} v_{j+1} \dots e_k v_k$ is a walk from v to w with strictly fewer edges than W. This contradictions the choice of W, so W has no repeated vertices. ■

Proof of Lemma 2

Restricting attention to the connected component of G that contains e, we can assume that G is connected.
First assume that e is in a cycle C in G. Then $C\backslash$e has two vertices x, y of degree 1 and the rest have degree 2.
$P: x = v_0 e_1 v_1 \dots e_k v_k = y$
To show that not a cut-edge, we show that $G\backslash$e is connect. Let $v, w \in V$. Since G is connected there is a walk In G from v to w. By lemma there is a path Q from v to w in G.
If Q does not use the edge e, then Q is a path in $G\backslash$e from v to w.
If Q uses e, then replace the edge e with the path P to get a walk from v to w in $G\backslash$e.
So there is also a path from v to w in $G\backslash$e. So $G\backslash$e is connected, so e is not a cut-edge.

Conversely, assume that e is not a cut-edge.
Then c($G\backslash$e) $= c(G)$ so vRw in G iff vRw in $G\backslash$e
Let $e = \{x, y\}$. Clearly xRy in G. Hence xRy in $G\backslash$e as well.
$x = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k = y$
Now $C = (\{v_0, v_1, \dots, v_k\}, \{e_1, e_2, \dots, e_k, e\})$ is a cycle containing edge e. ■

Examples of Minimally Connected Graphs

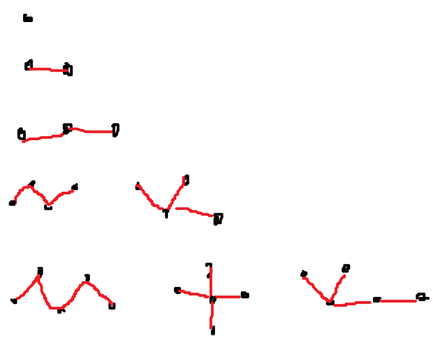
p=1

p=2

p=3

p=4

p=5



Trees

October-26-11 1:44 PM

Tree

A graph is a tree if it is connected and contains no cycles.

Lemma

Let T be a tree with $p \geq 2$ vertices. Then T has at least two vertices of degree 1.

Lemma

Let G be a graph and let $v \in V$ be a vertex of degree 1. Let $G \backslash$v be the subgraph of G spanned by $V \backslash \{v\}$

a) G is connected iff $G \backslash$v is connected
b) G contains a cycle iff $G \backslash$v contains a cycles.

Proof by observation

Proposition

Let T be a tree with p vertices and q edges. Then $q = p - 1$

Handshake Lemma

Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg_G v = 2q$$

Proof of Lemma

T is a connected graph with $p \geq 2$ vertices so T has $q \geq 1$ edge.
Let P be a path in T that is as long as possible. Then P has length ≥ 1, so the ends x, y of P are distinct: $x \neq y$

Claim

$\deg_T(x) = 1$
Then $\deg_T(y) = 1$ by symmetry

Suppose $\deg_T(x) \geq 1$. Let $P: v_0 e_1 v_1 e_2 \ldots e_k v_k = y$
Since e_1 is incident with x, there is another edge $f = \{x, z\} \in E$ incident with X.
Since P is as long as possible $zfxe_1v_1e_2 \ldots w_k v_k = y$ is not a path. It is a walk and has no repeated edges the only way it can fail to be a path is if $z \in \{v_2, \ldots, v_k\}$. This implies that T contains a cycle, a contradiction ■

Proof of Proposition

Induction on p.
Basis $p = 1$. T has 1 vertex and no edges. $\Rightarrow q = p - 1$

Induction: Assume holds for a tree with $p - 1$ vertices
$p \geq 2$. T has a vertex v of degree 1 by Lemma 1. By Lemma 2 $T \backslash$v is connected and contains no cycles $\Rightarrow T \backslash$v is a tree with $p - 1$ vertices. By induction hypothesis T with v deleted has $p - 2$ edges.
T with v deleted has 1 fewer vertiex, and 1 fewer edge so T has $(p - 2) + 1 = p - 1$ edges.

Proof of Handshake Lemma

Let X be the set of paris $X = \{(v, e) \in V \times E : v \in e\}$

$$|X| = \sum_{w \in V} |\{e \in E : w \in e\}| = \sum_{w \in V} \deg_G(w)$$

$$|X| = \sum_{f \in E} |\{v \in V : v \in f\}| = \sum_{f \in E} 2 = 2q$$

■

Spanning Trees

October-28-11 1:30 PM

Proposition

Let $G = (V, E)$, and $e = \{x, y\}$ a cut-edge of G.
Then $G\backslash$e has exactly 2 components X,Y with $x \in V(X), y \in V(Y)$

Let $c(G)$ be the number of connected components of G

Corollary 1

$c(G) \leq c(G\backslash e) \leq c(G) + 1$

Corollary 2

If G has p vertices and q edges then $c(G) \geq p - q$.

Corollary 3

If G is connected with p vertices and q edges then $q \geq p - 1$

The 2/3 Theorem (Trees)

Consider the following 3 conditions:

1) G is connected
2) G has no cycles
3) $q = p - 1$

Then any two of these implies the remaining one.

Spanning Subgraph

Let $G(V, E)$ be a graph. A subgraph $H(W, F)$ of G is spanning if $W = V$. That is, H uses all the vertices of G.

Spanning Tree

A spanning tree is a spanning subgraph of G that is a tree.

Proposition

G has a spanning tree iff G is connected.

Proof of Proposition

Let X be the component of $G\backslash$e containing x, an let Y be the component of $G\backslash$e containing y.
We need to show that $X \neq Y$ and every $z \in V$ is either in X or in Y.
First, suppose that X = Y. Then xRy in $G\backslash$e
Then there is a path P in $G\backslash$e from x to y
Now (V(P), $E(P) \cup \{e\}$) is a cycle in G containing e. Hence e is not a cut-edge of G; contradiction.

Secondly, let $z \in V(G)$. Since G is connected, there is a path Q in G from x to z. If Q does not use the edge e then xRz in $G\backslash$e so $z \in V(X)$ in this case.
If Q does use the edge e, then e is the first edge of Q (starting at x) since Q has no repeated vertices.
$Q: xey \dots e_k z$
The segment of Q from y to z is a path in $G\backslash$e from y to z, so yRz in $G\backslash$e, so $z \in V(Y)$
■

Proof of Corollary 2

Induction on q.
Basis: $q = 0$, G has p vertices, 0 edges, p components.
$c(G) = p - 0$ in this case.

Induction step, $q \geq 1$. Let $e \in E$
Then $c(G\backslash e) \leq c(G) + 1$
and $c(G\backslash e) \geq p - (q - 1)$ by induction so $c(G) \geq p - q$

Proof of Corollary 3

$1 \geq p - q$ by the previous corollary ■

Proof of 2/3 Theorem

1&2 ⇒3

Proved last lecture

1&3⇒ 2

Assume that G is connected and $q = p - 1$. Suppose that G has a cycle C. Let e be an edge in C.
Then e is not a cut-edge of G. So $G\backslash$e is connected with p vertices and $q = (p - 1) - 1 = p - 2$ edges.
This contradicts corollary 3

2&3 ⇒ 1

G has no cycles and $q(G) = p(G) - 1$
Let $G_1, G_2, \dots, G_c$ be the connected components of G and let G_i have p_i vertices and q_i edges. Each G_i is a connected graph with no cycles. Since 1&2 ⇒ 3 we have that $q_i = p_i - 1 \; \forall 1 \leq i \leq c$
Now $p(G) = p_1 + p_2 + \dots + p_c,\;\; q(G) = q_1 + q_2 + \dots + q_c$
$1 = p(G) - q(G) = (p_1 + \dots + p_c) - (q_1 + \dots + q_c) = (p_1 - q_1) + (p_2 - q_2) + \dots + (p_c - q_c) = c$
Since $c(G) = 1$, G is connected ■

Proof of Proposition

If G has a spanning tree T then G is connected, since T is connected and spanning. Conversely, assume that G is connected. Proceed by induction on $q(G)$

Basis: $q = p - 1$. This this case 2/3 theorem implies that G is a tree. So it is a spanning tree of itself.

Induction Step: $q > p - 1$. Then G has a cycle (otherwise it is a tree, and $q = p - 1$). Let e be an edge in a cycle of G. Then $G\backslash$e is still connected and has $q - 1$ edges. By induction $G\backslash$e has a spanning tree, which is also a spanning tree of G.

Search Trees

October-31-11 1:32 PM

Search Tree Algorithm

Let $G = (V, E)$ be a graph, and $v_0 \in V$ be a "base" vertex.
Initially, let $W = \{v_0\}$ and let $F = \emptyset$
*
Let Δ be the set of edges with one end in W and one end not in W.

If $\Delta = \emptyset$ then output (W, F) and stop.
If $\Delta \neq \emptyset$ then let $e = \{x, y\} \in \Delta$ with $x \in W$ and $y \notin W$
Update: $W \leftarrow W \cup \{y\}, \quad F \leftarrow F \cup \{e\}$ and goto *

Proposition

Let $G = (V, E)$ be a graph, v_0 a vertex of G, and let $T = (W, F)$ be output by an application of the search tree algorithm to G and v_0. Then T is a spanning tree for the connected component of G containing v_0

Note

Note that the search tree algorithm gives a path from any vertex to the base vertex.

Specialize search tree algorithm so that for each $w \in W$ the path from w to v_0 in T is a shortest path from w to v_0 in G

Length of a path

of edges of the path

Distance between vertices

The distance from vertex x to vertex y is the minimum length of any path from x to y. Denoted $dist_G(x, y)$

Breadth-First Search

Vertices in W are recorded in a queue.
Calculate Δ as before. If $\Delta \neq \emptyset$ let $e = \{x, y\} \in \Delta$ with $x \in W$ and $y \neq W$ and x as early in the queue as possible. y joins the end of the Δ queue.

$dist_T(a_0, z) = dist_G(a_0, z)$

Depth-First Search

Record the vertices in W in a stack.
Calculate Δ as before. Chose $e = \{x, y\} \in \Delta$ with x as close to top of the stack as possible. Add y to the top of the stack.

Proof of Proposition

(W, F) is a tree.
Induction on the number of iterations of the loop:

Basis of induction: $W = \{v_0\}, F = \emptyset$.
$(\{v_0\}, \emptyset)$ is connected and has no cycles - it is a tree.

Induction step: Assume that (W, F) is a tree.
$\Delta \neq \emptyset$ and $e = \{x, y\}$ and $W' = W \cup \{y\}, F' = F \cup \{e\}$
Since (W, F) is a tree, xRw in (W', F') for all $w \in W$
Also xRy since $e \in F'$ so $xRz\ \forall z \in W'$
So (W', F') is connected.
Let $|W| = p$ and $|F| = q$ so that $q = p - 1$ as (W, F) is a tree
Now $|W'| = p + 1$ and $|F'| = q + 1$ so $|F'| = |W'| - 1$

From these and the 2/3 algorithm we get that (W', F') is a tree.
End of induction, so (W, F) is a tree.

To see that (W, F) spans the component H of G containing v_0:
Since $v_0 Rw \forall w \in W$ (W, F) is a subgraph of H. Let z be any vector in H.
Suppose that $z \neq W$. Since $v_0 Rz$ in G there is a path P in G from v to z. Since $v_0 \in W$ and $z \notin W$ there is an edge f of P with one end in W and one end not in W.
But then $f \in \Delta$ so $\Delta \neq \emptyset$ so the algorithm has not terminated yet. Contradiction ∎

Breadth-First Search

November-02-11 1:38 PM

Notation

$G = (V, E)$ and $v \in V$ let $E(v)$ be the set of edges of G incident with v.
$E(v) = \{e \in E: v \in e\}$

Symmetric Difference

For sets A,B, the symmetric difference of A and B is $A \oplus B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ the set of elements in A or B but not both.

Breadth First Search

Input:

Graph $G = (V, E)$, vertex $v_0 \in V$

Initialize:

$W = \{v_0\}, \quad F = \emptyset, \quad \Delta = E(v_0)$
Put v_0 on front of queue Q.

While $\Delta \neq \emptyset$

Let v_i be the earliest vertex on Q such that $\Delta \cap E(v_i) \neq \emptyset$
Let $e = \{v_i, y\} \in \Delta \cap E(v_i)$ *so* $y \notin W$

Update:

$W \leftarrow W \cup \{y\}, \quad F \leftarrow F \cup \{e\}$
Put y on the end of Q
Level: $l(y) = l(v_i) + 1$
Parent: $pr(y) = v_i$
$\Delta \leftarrow \Delta \oplus E(y)$

Output $\left((W, F), l, pr\right)$

Eventual Claim

The path in $T = (W, F)$ from v to v_0 is a path in G from v to v_0 that is a short as possible.
That is, $dist_G(v, v_0) = l(v)$

Observation

1. When v joins the queue, earliest vertex on Q with $E(v_i) \cap \Delta \neq \emptyset$ is $pr(v)$
   Call v_i, the earliest vertex on the queue, the active vertex.

2. A vertex can become active, then stop being active, but then it never becomes active again.

3. If x occurs before y in Q (and neither one is v_0) then $pr(x)$ occurs before $pr(y)$ in Q or $pr(x) = pr(y)$.

4. If x occurs before y on Q then $l(x) \leq l(y)$

Proof of Observations

3rd Part

Suppose x occurs before y in Q but $pr(y)$ occurs before $pr(x)$
Since $pr(x)$ is active when x joins the queue $E\left(pr(y)\right) \cap \Delta = \emptyset$
By y joins Q after x so when x joins Q the edge $e = \{pr(y), y\}$ is in $E\left(pr(y)\right) \cap \Delta \neq \emptyset$. Contradiction ∎

3 => 2

The active vertex moves from left to right along Q.

4th

By induction on the positions of y in the queue since x occurs before y, $y \neq v_0$.
If $x = v_0$ then $0 = l(v_0) = l(x) \leq l(y)$
So assume that $x \neq v_0$
Now by 3 $pr(x)$ occurs before $pr(y)$ on Q. By induction $l\left(pr(x)\right) \leq l\left(pr(y)\right)$
So $l(x) = l\left(pr(x)\right) + 1 \leq l\left(pr(y)\right) + 1 = l(y)$
∎

Distance in Graphs

November-04-11 1:33 PM

Construct a Breadth First Search Tree

- $pr(x)$ is active when x joins the queue
- If x occurs before y on the queue then $pr(x)$ occurs before $pr(y)$ in Q
- The active vertex moves left to right in Q
- The level of vertices increases from left to right on Q.

Fundamental Property of BFS

Let $G = (V, E)$ be a connected graph. Let T be a breadth first search tree for G. Let $l_T(v)$ be the level of $v \in V$ in T.

Let $e = \{x, y\} \in E$ be any edge of G. Then $|l_T(x) - l_T(y)| \leq 1$

Note:

Not true for search trees in general.

Theorem

Let $G = (V, E)$ be a connected graph, $v_0 \in V,$ and let T be a BFST for G with base vertex v_0 them for every $v \in V$
$dist_G(v, v_0) = l_T(v)$

Facility Location Problem

Measure of v

$$f(v) = \sum_{w \in v} dist_G(v, w)$$

Find a vertex that minimizes $f(v)$

Algorithm

For each $v \in V$:

- Compute a BFST T for G based at v
- $f(v) = \sum_{w \in V} l_T(w)$

Computed Girth

For each $v \in V$ grow a BFST T of G based at v
For each edge $e = \{x, y\}$ in G but not in T let $m(e) = l_T(x) + l_T(y) + 1$
Let $g(v) = \min_{e \in G \setminus T} m(e)$
$Let\ \gamma = \min_{v \in V} g(v)$

Claim

γ is the girth of G

Correctness of this algorithms depends on if C is a cycle in G that is as short as possible and v is a vertex in C then $g(v)$ is the length of C.

Test of Bipartness

Input a connected graph $G = (V, E)$. Grow a BFST based at any $v_0 \in V$.
G is bipartite iff for every $e = \{x, y\} \in E$ $|l_T(x) - l_T(y)| = 1$
By partition: (even level, odd level)

Diameter of a Graph

$diam(G) = \max_{v,w \in V} dist_G(v, w)$

Proof of Fundamental Property of BFS

If $e = \{x, y\}$ is in T then either $x = pr(y)$ or $y = pr(x)$ so
$l_T(x) = l_T(y) - 1$ or $l_T(x) = l_T(y) + 1$

Suppose that $\left|l_T(x) - l_{t(y)}\right| \geq 2$
Assume that $l_t(x) \leq l_T(y) - 2$
So $pr(x), x, pr(y), y$ occur in that order on Q (since $l_T(x)$ is weakly increasing from left to right.)
$pr(y)$ is active when y joins the queue, so $E(x) \cap \Delta = \emptyset$ when y joins the queue. But $e = \{x, y\} \in E(x) \cap \Delta$ when y joins the queue.

Proof of Theorem

The unique path in T from v to v_0 has $l_T(v)$ edges.
Thus $dist_G(v, v_0) \leq l_T(v)$
Conversely, let P be any path in G from v to v_0
$P: v = z_0 e_1 z_1 e_2 z_2 \dots z_{k-1} e_k z_k = v_0, say\ P\ has\ k\ edges$

$$l_T(v) = l_T(v) - l_T(v) = \sum_{k=1}^{k} |l_T(z_{i-1}) - l_T(z_i)| \leq \sum_{i=1}^{k} 1 = k$$

So every path from v to v_0 has at least $l_T(v)$ edges.
So $dist_G(v, v_0) = l_T$

Planar Graphs

November-07-11 1:30 PM

Graphs which can be drawn without crossing edges.

Planar Embedding

Let $G = (V, E)$ be a graph.
A **plane embedding** of G is a pair $\{p_v : v \in V\}$ and $\{\gamma_e : e \in E\}$ whose

- p_v are pairwise distinct points in $\mathbb{R}^2$ (if $v \neq w$ *then* $p_v \neq p_w$) and
- γ_e are simple curves in $\mathbb{R}^2$ (image of $[0,1]$ under some continuous function $f: [0,1] \to \mathbb{R}^2$ that is injective) i.e. γ_e does not intersect itself and
- if $e = \{x, y\} \in E$ then γ_e has end points p_x and p_y and
- If $\gamma_e \cap \gamma_f \neq \emptyset$ then both e and f are incident with a common vertex w and $\gamma_e \cap \gamma_f = \{p_w\}$

γ_e are images of functions (the set of points corresponding to the curve in $\mathbb{R}^2$

Planar Graph

A planar graph is a graph that has some plane embedding.

Faces

Let $\{p_v : v \in V\}$ and $\{\gamma_e : e \in E\}$ be a plane embedding of a graph $G = (V, E)$.
The **faces** of the embedding are the connected components of

$$\mathbb{R}^2 \setminus \bigcup_{e \in E} \gamma_e$$

Degree of a Face

The **degree** of a face is the number of edges on its boundary counted with multiplicities.

E.g.

The embeddings drawn for 'two plane embeddings' have 4 faces each.

Handshake Lemma for Faces

Let G be a graph property embedded in the plane, with q edges

$$\sum_{F: a\ face} \deg(F) = 2q$$

Proposition

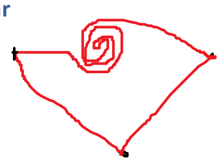
Let $G = (V, E)$ be a plane embedding. Let $e \in E$ and let the faces with e on their boundaries be F_1 and F_2. Then $F_1 = F_2$ iff e is a cut-edge.

Euler's Formula

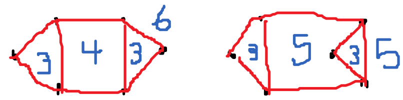
Let G be a plane graph with p vertices, q edges, r faces, and c connected components.
Then $p - q + r = c + 1$

Not Planar

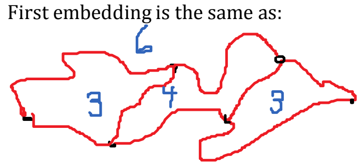
Planar



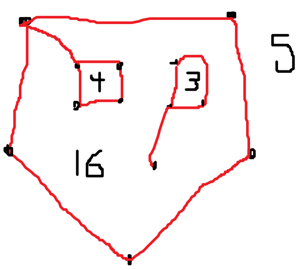
Two plane embeddings of the same graph



First embedding is the same as:



Degree of Faces Example



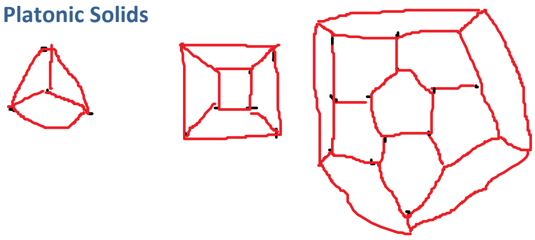
Proof of Proposition

If e is not a cut-edge then e is contained in a cycle C.

Then $\bigcup_{f \in E(C)} \gamma_f$ separates F_1 *from* F_2 so $F_1 \neq F_2$

Conversely, if $F_1 \neq F_2$ then walk around F_1 starting and ending at the edge e - you get a closed walk containing e. Deleting subwalks between repeated vertices produces a cycle containing e. So e is not a cut-edge.

Platonic Solids



p	4	8	6	20	12
q	6	12	12	30	30
r	4	6	8	12	20

Proof of Euler's Formula

Induction on q:
Basis: $q = 0$ Then $r = 1$ and so
$p - 1 + r = p + 1 = c + 1$. Good

Induction step:
Let $e \in E$ and consider $G' = G \setminus e$ with p', q', r', c' vertices, edges, faces, and components.

If e is a cut-edge then
$p = p, \quad q = q' + 1, \quad r = r', c = c' - 1$

$p - q + r = p' - (q' + 1) + r' = (p' - q' + r') - 1 = c' + 1 - 1$
$= c' = c + 1$

If e is not a cut-edge then
$p = p', \quad q = q' + 1, \quad r = r' + 1, \quad c = c'$
$p - q + r = c + 1$

Condition for Embedding

November-09-11 1:32 PM

Euler's Formula

Let G be embedded in $\mathbb{R}^2$ with p vertices, q edges, r faces, and c components.
Then $p - q + r = c + 1$

Corollary

Let G be a graph with p vertices and $q \geq 2$ edges. If G is planar then
$q \leq 3p - 6$

Note of Exception

If $q = 1, p = 2$: $1 \nleq 3 \times 2 - 6$
If $q = 0, p = 1$: $0 \nleq 3 \times 1 - 6$

Corollary

Let G be a bipartite graph with p vertices and $q \geq 2$ edges. If G is planar then
$q \leq 2p - 4$

Subdivision

Subdivision of an edge $e = \{x, y\}$ in a graph $G = (V, E)$
This is the graph $G \cdot e$ with vertex-set $V' = V \cup \{z\}$ where $z \notin V$ and edge set $E' = (E \setminus \{e\}) \cup \{\{x, z\}, \{y, z\}\}$

Claim

G is planar iff $G \cdot e$ is planar.
Exercise

Two graphs related by a finite sequence of subdivisions or reverse subdivisions are either both planar or both not planar

Lemma

If H is a subgraph of G and G is planar then H is planar.

Corollary

Any graph that contains a (repeated) subdivision of K_5 or $K_{3,3}$ is not planar.

Kuratowski's Theorem

A graph is planar iff it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Proof

CO 342

Proof of Corollary

Consider any plane embedding of G, with r faces. Since $q \geq 2$ every face of the embedding has degree ≥ 3.
By the Handshake Lemma for faces:

$$2q = \sum_{face\ F} \deg(F) \geq 3r$$

Since $q \geq 2, p \geq 1$ so $c \geq 1$ by Euler's Formula
$p - q + r = c + 1 \geq 2$
$3p - 3q + 3r \geq 6$
$3p - 3q + 2q \geq 3p - 3q + 3r \geq 6$
$3p - q \geq 6$ so $q \leq 3p - 6$ ∎

Proof of Corollary

Consider any plane embedding of G with r faces
Since $q \geq 2$ and G is bipartite, every face has degree ≥ 4
By Handshake lemma for faces, $2q \geq 4r \Rightarrow q \geq 2r$
Since $q \geq 2, p \geq 1, so\ c \geq 1$
$p - q + r \geq 2$
$2p - 2q + 2r \geq 4$
$2p - 2q + q \geq 4$
$q \leq 2p - 4$

Numerology for Planar Graphs

November-11-11 1:32 PM

Vertex Degrees in a Planar Graph

Planar graph, p vertices, q edges $(q \geq 2)$, n_k vertices of degree k $(k \geq 0)$

Then $q \leq 3p - 6$
$p = n_0 + n_1 + n_2 + \cdots + n_{p-1}$
$$2q = \sum_k kn_k$$
$$2q \leq 6p - 12 \Rightarrow \sum_k kn_k \leq \sum_k 6n_k - 12$$
$$12 \leq \sum_{k=0}^{p-1} (6-k)n_k$$
$\Rightarrow 12 \leq 6n_0 + 5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 - n_7 - 2n_8 - 3n_9 - \cdots$
$n_5 + 2n_4 + 3n_3 + 4n_2 + 5n_1 + 6n_0 \geq 12 + n_7 + 2n_8 + 3n_9 + \cdots$

In a planar graph of minimum degree ≥ 2
$n_5 + 2n_4 + 3n_3 + 4n_2 \geq 12$
In a simple planar graph there must be a vertex of degree ≤ 5

The Four-Colour Theorem

Conjecture made in 1851 by Guthrie

For any plane graph, the faces can be coloured with a most four colours so that neighbouring faces get different colours.
Proved in 1974 by Appel and Haken.

Planar Duality

G is a plane graph
G* is its dual graph.
Draw one vertex of G* on each face of G. Draw one edge of G* across each edge of G

With this can end up with duplicate edges, or edges back to the same vertex.

Multigraph

$G = (V, E)$
V: set of vertices
E: multiset of 2 element multisubsets of V
e.g. $G = (\{1,2,3\}, \{\{1,1\}, \{2,3\}, \{2,3\}, \{1,2\}, \{2,2\}, \{2,2\}\})$

Proposition

G* can be drawn on G without any edges of G* crossing.

Proposition

$(G^*)^* = G$

Four Colour Theorem

Let G be a planar multigraph without loops. Then $V(G)$ can be coloured with ≤ 4 colours so that adjacent vertices get different colours.

$\chi(G) \leq 4$

Proper k-Colouring

Leg $G = (V, E)$ be a multigraph proper k-colouring.
$f: V \to \{1, 2, \ldots, k\}$ such that if $\{v, w\} \in E$ then $f(v) \neq f(w)$.

Chromatic Number

The chromatic number of G is
$\chi(G) = \min\{k : G \text{ has a proper k-colouring}\}$

Spherical Projections

A graph can be drawn on a plane iff it can be drawn on a sphere.
You just need to avoid the north pole.

Exercise

$p \geq 3$ vertices, q edges, c components
No faces of degree 3

a) $q \leq 2p - 4c$
b) Phrase this in terms of n_k

Proof of Proposition

By induction on $q = |E(G)|$

Basis

$q = 0$ is trivial

Induction

If every edge of G is a cut-edge then G has no cycles, so it has only one face. G* has one vertex, and one loop for each edge of G. Loops can be drawn without overlap.

If e is not a cut-edge of G then consider G\e and (G\e)* By induction can draw (G\e)* without crossing edges. Can add in e without crossing.

Alternately

Put a vertex in each face. Can draw a half-edge to each edge of that face in G. Connect those half-edges at the edges of the faces and have no crossings.

G and G* are both embedded in the plane. Edge e of G meets edge f* of G* if and only if e=f in which case $e \cap e^*$ is a single point.

Colour Theorems

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Note

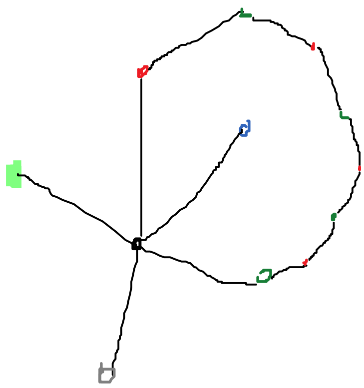
$\chi(G) \leq 2$ iff G is bipartite.
$\chi(G) \leq 1$ iff G has no edges
$\chi(G) = 0$ iff G has no vertices

Six Colour Theorem

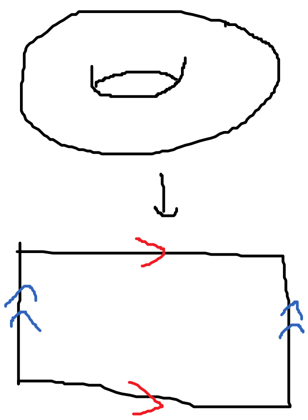
If G is a planar graph then $\chi(G) \leq 6$

Five Colour Theorem

If G is a planar graph then $\chi(G) \leq 5$



Graphs on Surfaces



Proof of The Six Colour Theorem

Induction on p, the number of vertices.

Base:
If $p \leq 6$ then give every vertex a different colour.

Induction:
Let G be planar with p vertices. G has a vertex of degree 5 or less, let v be such a vertex.

By induction, $G\backslash v$ has a proper six-colouring $f: V\backslash e \to \{1, 2, \dots, 6\}$
Let the neighbours of v b $z_1, \dots, z_k$ where $k \leq 5$. $\{f(z_1), \dots, f(z_k)\}$ has at most 5 colours.
$\exists c \in \{1, \dots, 6\}$ such that $c \notin \{f(z_1), \dots, f(z_k)\}$ and set $f(v) = c$

Proof of the Five Colour Theorem

Induction on $p = |V(G)|$

Base
$p \leq 5$: give every vertex a different colour.

Induction Step:
Let G be planar with p vertices. Let $v \in V$ have degree ≤ 5.
Let $f: V \setminus \{v\} \to \{1, 2, 3, 4, 5\}$ be a proper 5 colouring of $G\backslash$v.
Let the neighbours of v be $z_1, \dots, z_k$ and let $S = \{f(z_1), \dots, f(z_k)\}$
If $S \neq \{1, 2, 3, 4, 5\}$ then $\exists c \in \{1, 2, 3, 4, 5\} \setminus S$ and we can set $f(v) = c$ to get a proper 5-colouring of G.

Remaining case: $S = \{1, 2, 3, 4, 5\}$
So v has 5 neighbours z_1, z_2, z_3, z_4, z_5. We can assume that G is embedded in the plane. WLOG $z_1, \dots, z_5$ occur in that order clockwise around v. Can also assume that $f(z_i) = i$

For $\{i, j\} \subseteq \{1,2,3,4,5\}$ let H_{ij} be the subgraph of $G\backslash v$ induced by the set of vertices coloured either i or j by f. If K is a connected component of H_{ij} then one can define a new 5-colouring of $G\backslash$v as follows:

For every $w \in V\backslash\{v\}$, $$g(w) = \begin{cases} f(w), & w \notin V(K) \\ i, & w \in V(K) \text{ and } f(w) = j \\ j, & w \in V(K) \text{ and } f(w) = i \end{cases}$$
Check: g is a proper 5-colouring of $G\backslash$v

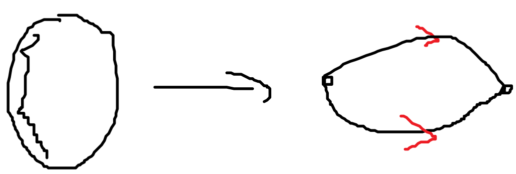
If z_1 and z_3 are in different components of H_{13} then let K be the component of H_{13} containing z_3.
Switch colours 3 and 1 on K to get g. Then $g(z_3) = g(z_1) = 1$
So we can set $g(v) = 3$ to get a proper 5-colouring of G.

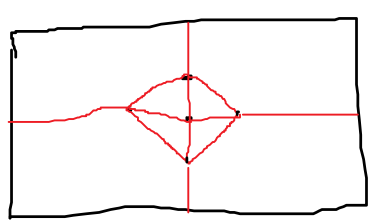
If z_1 and z_3 are in the same connected component of H_{13} then there is a path in $G\backslash$v from z_1 to z_3 in which every vertex is coloured 1 or 3 by f.

Since G is planar the path P with edges $\{v, z_1\}, \{v, z_3\}$ forms a cycle that separates z_2 from z_4. Thus z_2 and z_4 are in different connected components of H_{24}. Recolour the component of H_{24} that contains z_4 and then give v colour 4. ■

Surfaces

Torus = rectangle with opposite sides identified





K5

Graphs on Surfaces

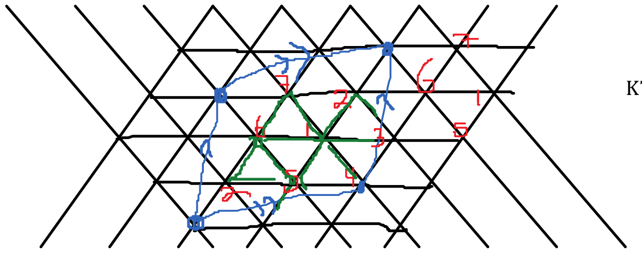
November-16-11 1:32 PM

Every graph can be embedded on some surface.
You can add loops for every vertex.

For any surface, there are finitely many obstructions to embedding a graph on that surface. It is hard to determine the surface with the fewest number of holes which allows a given graph to be embedded.

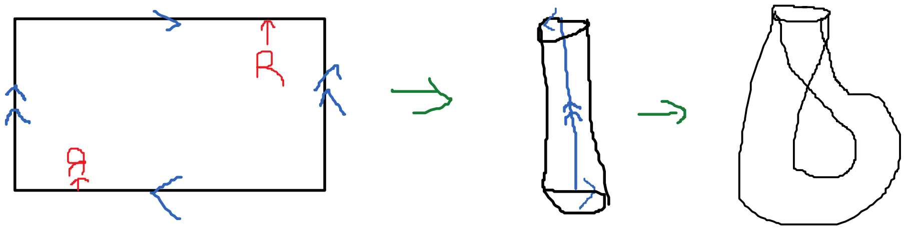
Surface Representations

Every surface can be represented (possibly non-uniquely) by a polygon with pairs of sides identified with each other.



K7 on the torus

Klein Bottle



This is a non-orientable surface. There is no distinction between clockwise and counter clockwise.

Non-orientable surfaces cannot be embedded in 3 dimensions, require at least 4.

Matching Theory

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Matching

Let $G = (V, E)$ be a graph. A matching, M, is a set of edges so that (V, M) has maximum degree ≤ 1.
Every vertex is in at most one edge of M.

Problem

Given G, find a matching on G of maximum size.

Perfect

A matching is perfect if every vertex has degree 1 in (V, M)

Non-Perfect Matching

A 2 regular graph consisting of an odd cycle has no perfect matching.

"Let's consider the next value of 2, which is 3."

M-Saturated

$v \in V$ is M-saturated if v is on an edge of M
$v \in V$ is M-unsaturated if v is not on any edge of M.

M-Alternating, M-Augmenting

Let $G = (V, E)$ be a graph.
M a matching of G
P a path in G, $p: v_0 e_1 v_0 \dots v_{k-1} e_k v_k$ is **M-alternating** if either
$e_i \in M \Leftrightarrow i$ is odd or
$e_i \in M \Leftrightarrow i$ is even

P is **M-augmenting** iff
- $e_i \in M \Leftrightarrow i$ is even, and
- P has an odd number of edges, and
- v_0 and v_k are M-unsaturated

Proposition

If M is a matching in G and P is an M-augmenting path then $M' = M \oplus E(P)$ is a matching in G with one more edge than M.
$S \oplus T = (S \cup T) \backslash (S \cap T)$

Theorem

Let $G = (V, E)$ be a graph. $\mathrm{M} \subseteq E$ a matching. Then M is a maximum matching iff G does not have an M-augmenting path.

Vertex Cover

A vertex cover is a set $S \subseteq V$ such that every edge $e \in E$ has at least one end in S.

Matching	Vertex Cover
Set of edges M	Set of vertices S
Every $v \in V$ is on ≤ 1 $e \in M$	Every $v \in V$ is on ≥ 1 $e \in M$
Find a maximum matching	Find a minimum vertex cover

Proposition

Let G be a graph, M a matching, and S a vertex cover in G.
Then $|M| \leq |S|$

Example: Odd Cycle

$\max |M| = \left\lfloor \frac{n}{2} \right\rfloor$

$\min |S| = \left\lceil \frac{n}{2} \right\rceil$

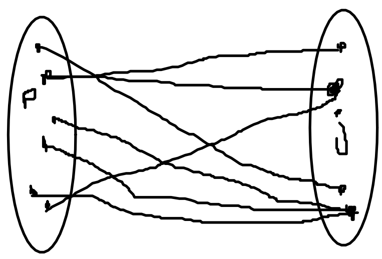
Corollary

Let G be a graph, M a matching, S a vertex cover.
If $|M| = |S|$ then M is a maximum matching and S is a minimum vertex-cover.

For a non-bipartite graph, there may be a gap, as in odd cycles (but not necessarily).

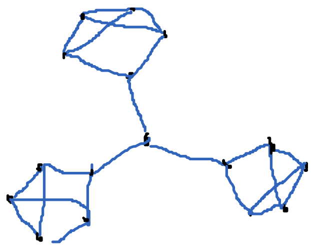
Toy Application

Processors Jobs

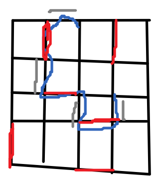


$\{p, j\}$ is an edge when processors in p can perform job j
Assign jobs to processors to maximize the number of busy processors.
$\leq$ one job per processor
$\leq$ one processor per job

3-Regular with no Perfect Matching



Example



Red are vertices in M, terminate on M-saturated vertices.
Blue is an M-augmenting path

Proof of Theorem

If P is an M-augmenting path in G, then $M' = M \oplus E(P)$ is a matching on G with $|M'| = 1 + |M|$ so M is not a maximum matching.

Conversely, assume that M is not a maximum matching. Let M^* be a maximum matching in G, so $|M^*| > |M|$
Consider the spanning subgraph (uses all the vertices) H of G with edges $M \cup M^*$.
In H, every vertex has degree 0, 1, or 2. Every connected component is either a path or a cycle. The cycles all have even length. Since $|M^*| > |M|$, there is a component K of H that has more edges in M^* than in M. Since connected components alternate 1 edge in M with 1 edge in M* this cannot be a cycle. This connected component must be a path with both end edges in M* but not in M.
The end vertices of K are not saturated by M. Thus K is an M-augmenting path.
■

Proof of Proposition

Let $X = \{(v, e) : v \in S, e \in M \text{ and } v \in e\}$
Since M is a matching, every $v \in S$ is in at most one $e \in M$ so

$$|X| = \sum_{v \in S} \sum_{e \in M} \begin{cases} 1, & v \in e \\ 0, & v \notin e \end{cases} \leq \sum_{v \in S} 1 = |S|$$

Since S is a vertex cover, every $e \in M$ is incident with at least one $v \in S$

$$|X| = \sum_{e \in M} \sum_{v \in S} \begin{cases} 1, & v \in e \\ 0, & v \notin e \end{cases} \geq \sum_{e \in M} 1 = |M|$$

So $|M| \leq |X| \leq |S|$

König's Theorem

November-21-11 1:55 PM

König's Theorem

Let G be a bipartite graph.
Then $\max|M| = \min|S|$
(Maximum over matchings M of G, minimum over vertex-covers S of G)

Algorithmification of König's Theorem

How to compute a maximum matching in a bipartite graph.

Input: a graph G with bipartition (A, B).
Initialize: $M = \emptyset$
Computation:

- Compute the set $X \subseteq A, Y \subseteq B$ as in Claims 1,2,3.
- If $y \in Y$ is M-unsaturated, find an M-alternating path P from some $x_0 \in X$ to y.
- Update $M \leftarrow M \oplus E(P)$,
- Repeat until there are no more M-unsaturated $y \in Y$.

Output: $(M, Y \cup (A \backslash X))$

Computing the sets X, Y systematically.

Input:

- Graph G with bipartition (A, B)
- Matching M in G

Initialize:

- X_0 to the M-unsaturated vertices in A.
- Put all vertices in X_0 on the front of queue Q.
- $X = X_0, Y = \emptyset$

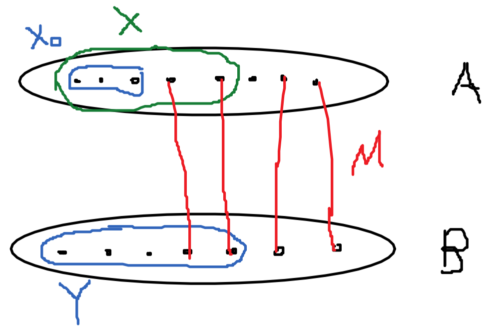
Computation:
While $Q \neq \emptyset$ do the following:

- Let q be the first vertex in Q
- If $q \in B$ and M-saturated then let $\{q, x\} \in M$, put x at the end of Q if x is not already in A. Delete q from the front of Q.
  $X \leftarrow X \cup \{x\}$
- If $q \in B$ and M-unsaturated then use q to find any M-augmenting path.
- If $q \in A$ then choose any non-matching edge $e = \{q, b\}$ with b not already on the Q. Adjoin b to the end of the Q. If there is no such b, delete q from the front of Q.
  $Y \leftarrow Y \cup \{b\}$

Output: (X, Y)

Anatomy of a Matching in a Bipartite Graph

Let G have bipartition (A, B)
Let M be a matching in G
Let $X_0 \subseteq A$ be the set of M-unsaturated vertices in A.
Let $X \subseteq A$ be the set of vertices reachable from some $x_0 \in X_0$ by an M-alternating path.
Let $Y \subseteq B$ be the set of vertices in B reachable from some $x_0 \in X_0$ by an M-alternating path.



Claim 1

If there is an M-unsaturated vertex $y \in Y$ then G has an M-augmenting path from some $x_0 \in X_0$ to y.

Proof

Let $x_0 \in X_0$ and let P be an M-alternating path from x_0 to y in G. Since neither x_0 nor y is saturated by M ($and\ x_0 \neq y$) P is an M-augmenting path.

Claim 2

there are no edges of G between the sets X and $B \backslash$Y

Proof

Suppose that $e = \{x, b\}$ with $x \in X$ and $b \in B$.
If $e \notin M$ then consider an M-alternating path P from some $x_0 \in X_0$ to $x \in X$. Then Peb is an M-alternating path from x_0 to b, so $b \in Y$ (since the last edge in P is in M)

If $e \in M$ then consider an M-alternating path P from some $x_0 \in X_0$ to $x \in X$.
$P: x_0 e_1 x_1 \dots x_{k-1} e_k x_k = x$. P has an even number of edges, $e_1 \notin M$ so $e_k \in M$, e_k is the unique matching edge on x. So $e_k = e$ and $y = x_{k-1} \in Y$.

Claim 3

There are no edges of M between the sets Y and $A \backslash$X.

Proof

Suppose that $e = \{a, y\}$ with $y \in Y$ and $a \in A \backslash$X.
Let P be an M-alternating path from x_0 to y. Then Pea is an M-alternating path from x_0 to a. So $a \in X$, a contradiction.

König's Theorem

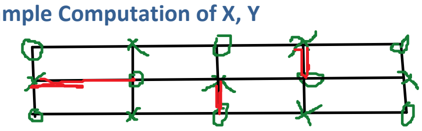
Let G be a bipartite graph. Let M be a maximum matching. Let S be a minimum vertex-cover.
Then $|M| = |S|$

Proof

Let M be a maximum matching in G and constructs sets X, Y as in claim 1,2,3.
Since M is a maximum matching, there are no augmenting paths.
By Claim 1, every vertex in Y is saturated by M.
By Claims 2, 3 every edge of M with one end in Y has its other end in X, and every edge of M with one end in $A \backslash$X as other end in $B \backslash$Y.
Every vertex in $(A \backslash \mathrm{X}) \cup Y$ is M-saturated. Now $|M| = |S|$ with $S = (A \backslash \mathrm{X}) \cup Y$. (Since each edge has one adjacent vertex in S)
By Claim 2, S is a vertex cover of G (since G has no edges between X and $B \backslash$Y, which are the only sets of M-unsaturated vertices.)

Hence S is a minimum size vertex-cover and $|S| = |M|$ ■

Example Computation of X, Y



Hall/Tutte Conditions

November-23-11 1:31 PM

A-Saturating
Let $G = (V, E)$ be a graph with bipartition (A, B). A matching M is A-saturating when every $a \in A$ is saturated by M.

Hall Condition
If G has an A-saturating matching M this defines an injective function $f: A \to B$ by saying that $f(a) = b$ iff $\{a, b\} \in M$.
If this exists then for all $S \subseteq A$, f restricts to an injective function from S to $N(S)$.

Thus, if G has an A-saturating matching then $|S| \leq |N(S)|\ \forall S \subseteq A$

Hall's Matching Theorem
Let $G = (V, E)$ be a graph with bipartition (A, B). Then G has an A-saturating matching iff $|S| \leq |N(S)| \forall S \subseteq A$.

Corollary
Let G be a k-regular graph with bipartition (A, B). If $k \geq 1$ then then G has a perfect matching.

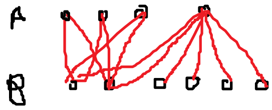
Corollary
A k-regular bipartite graph can be partitioned into k edge-disjoint perfect matching.

Tutte Condition
Let $G = (V, E)$ be a graph.
For $S \subseteq V$ let $G\backslash S$ be the subgraph of G induced by vertices in $V\backslash S$.
Let odd$(G\backslash S)$ be the number of connected components of $G\backslash S$ with an odd number of vertices.

If G has a perfect matching then for every $S \subseteq V, |S| \geq odd(G\backslash S)$.

Tutte's Matching Theorem
A graph has a perfect matching iff
$\forall S \subseteq V, \quad |S| \geq odd(G\backslash S)$

Which bipartite graphs have A-saturating matchings?



Does not have an A-saturating matching.

For each $S \subseteq A$, let $N(S) = \{b \in B: \{a, b\} \in E$ for *some* $a \in S\}$
This example has a set $S \subseteq A$ with $|S| = 3$ and $|N(S)| = 2$
If G has an A-saturating matching M this defines an injective function $f: A \to B$ by saying that $f(a) = b$ iff $\{a, b\} \in M$.

Proof
We've seen that if G has an A-saturating matching then $\forall S \subseteq A: |S| \leq |N(S)|$
Conversely, assume that there is no A-saturating matching. Let M^* be a maximum matching in G. So $|M^*| < |A|$.
By König's Theorem, there is a vertex-cover Q in G with $|Q| = |M^*|$.
Since Q is a vertex cover, there are no edges from $S = A\backslash Q$ to $B\backslash Q$
In other words, $N(S) \subseteq Q \cap B$
$|Q \cap A| + |Q \cap B| = |Q| = |M^*| < |A|$
$|A| - |Q \cap A| > |Q \cap B|$
$|S| = |A\backslash Q| = |A| - |Q \cap A| > |Q \cap B| \geq |N(S)| \Rightarrow |S| > |N(S)|$ ■

Proof of Corollary
Since $k \geq 1$ we have $|A| \times k = q = |B| \times k$ so $|A| = |B|$
So every A-saturating matching is also a B-saturating matching.

Check Hall's Conditions
Let $S \subseteq A$ and consider $N(S)$. Counting edges of G with one end in S we get $k|S| \leq k|N(S)|$.
By Hall's Theorem there is an A-saturating matching.

Proof of Tutte's Condition
On homework

Problem
Consider a bipartite graph that is biregular. There are integers $a \geq 0, b \geq 0$ such that every vertex in A has degree a and every vertex in B has degree b.
Assume that $\gcd(a, b) = d$ and write $a = da'$ and $b = db'$.

Does G have a spanning subgraph that is (a', b') biregular?
Yes, true for all a and b.

Example: $a = 4, b = 2$
Note that when $a = b, d = a = b, a' = b' = 1$ and (a', b') biregular subgraph is a perfect matching.

Counting Spanning Trees

November-25-11 1:31 PM

Notation

$\kappa(G)$ is the number of spanning trees of G
$G\backslash e$ G **delete** e
G/e G **contract** e
"Shrink" the edge until the ends of it merge intro a single vertex. Produces a multigraph.

Deletion-Contraction Recurrence

For any graph G and $e \in Ep$
$\kappa(g) = \kappa(G\backslash \text{e}) + \kappa(G/e)$

Cut-Vertex

A cut vertex is a vertex which, when deleted, increases the number of connected components in the graph.

If G has a cut-vertex v Then let $G_1, \ldots G_c$ be the components of $G\backslash$v each with v joined back in. Then

$$\kappa(G) = \prod_{i=1}^{c} \kappa(G_i)$$

Cycle

The number of spanning trees for an n-cycle is n.
This is true even for cycles of length 1 or 2.

Adjacency Matrix

The adjacency matrix $G = (V, E)$ A, indexed by $V \times V$

$$A_{v,w} = \begin{cases} 1 \; if \; \{v,w\} \in E \\ 0 \; if \; \{v,w\} \notin E \end{cases}$$

more generally for multigraphs:

$$A_{v,w} = \begin{cases} \#\text{edges joining v and w if } v \neq w \\ \quad 2 \times \#\text{ loops at v } if \; w = v \end{cases}$$

Δ square diagonal matrix indexed by $V \times V$

$$\Delta_{v,w} = \begin{cases} \quad 0 \; if \; v \neq w \\ \deg_G(v) \; \; if \; v = w \end{cases}$$

Laplacian Matrix

$L = \Delta - A$

Matrix-Tree Theorem

Let $v \in V$ be any vertex and let $L(v|v)$ be obtained by deleting row v and column v of L.

$\kappa(G) = \det L(v|v)$

Signed Incidence Matrix

Let $G = (V, E)$ be a connected multigraph
Draw an arrow on each edge $\{v, w\}$ in an arbitrary direction, either $v \to w$ or $w \to v$

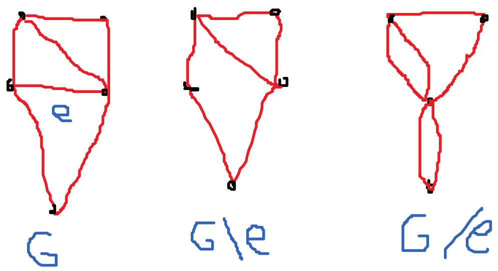
D is indexed by $V \times E$

$$D_{v,e} = \begin{cases} +1 \text{ if } e \text{ points into } v \text{ but not out} \\ -1 \text{ if } e \text{ points out of } v \text{ but not in} \\ \qquad 0 \text{ otherwise} \end{cases}$$

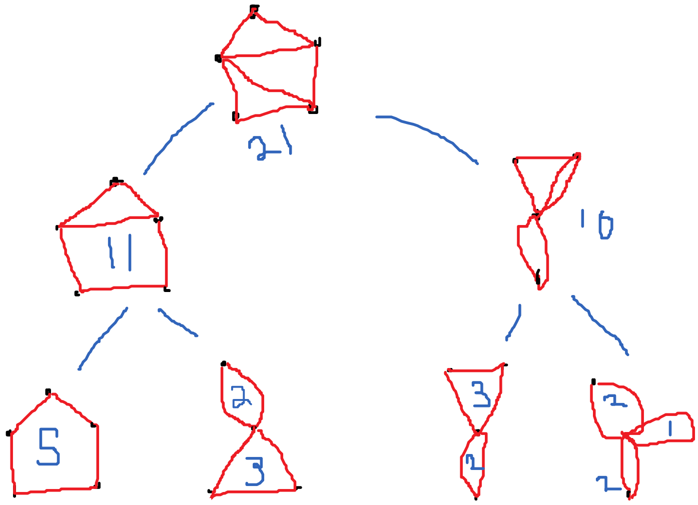
Fact

For any orientation of G
$DD^T = \Delta - A$

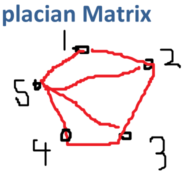
Contracting, Deleting



Example of Deletion-Contraction Recurrence



Example of Laplacian Matrix

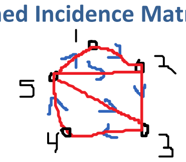


$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

$$\det L(5|5) = \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 2\big(3|6-1| + (-2)\big) - (6-1) = 26 - 6 + 1 = 21$$

Example of Signed Incidence Matrix



$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$$DD^T = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} = L(G) = \Delta - A$$

Matrix Tree Theorem

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$G = (V, E)$ a connected multigraph
A **adjacency matrix** indexed by $V \times V$

$$A_{v,w} = \begin{cases} \#\text{ edges with ends } \{v, w\}, & v \neq w \\ 2 \times \#\text{ loops at } v, & v = w \end{cases}$$

Degree matrix diagonal $V \times V$
$\Delta_{v,v} = \deg_G(v)$

Laplacian matrix: $L(G) = \Delta - A$

D is a $V \times E$ **signed incidence matrix** for G with respect to an arbitrary orientation of G

$$D_{v,e} = \begin{cases} +1 \text{ if } e \text{ points into } v \text{ but not out} \\ -1 \text{ if } e \text{ points out of } v \text{ but not in} \\ 0 \text{ otherwise} \end{cases}$$

$L(G) = \Delta - A = DD^T$ if G has no loops

Matrix-Tree Theorem

For any vertex $w \in V$, $\kappa(G) = \det L(w|w)$

The Binet-Cauchy Identity

Let M be an $r \times m$ matrix and P be an $m \times r$ matrix. Then

$$\det(MP) = \sum_S \det(M(|S]) \cdot \det(P[S|))$$

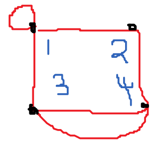
with summation over all r-element subsets $S \subseteq \{1, 2, \dots, m\}$

For a matrix Q and sets I, J of row and column indices,
$Q[I|J]$ is the submatrix of Q indexed by rows $i \in I$ and columns $j \in J$.
$Q(I|J)$ is the submatrix of Q indexed by rows $i \notin I$ and columns $j \notin J$
$M(|S]$ means delete no rows, keep only columns in S

Proposition

Let $G = (V, E)$ be a connected multigraph. Let $R \subseteq V$ and $S \subseteq E$ be such that $|R| + |S| = |V|$ and $R \neq \emptyset$
Consider $D(R|S]$.
Then $det\ D(R|S]{=}\pm 1\ iff\ (V,S)$ is a forest has a unique vertex in R
$and\ \ det\ D(\mathrm{R}|\ \mathrm{S}]{=}0$ if not.

Example Laplacian Matrix



$$T = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 3 & -2 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

Setup of Matrix-Tree Theorem Proof

Since $L = \Delta - A = DD^T$ use Binet-Cauchy

$$\det L(w|w) = \det DD^T(w|w) = \det D(w|\square)D^T(\square|w) = \sum_S \det D(w|S] \cdot \det D^T[S|w)$$

Summation over all sets $S \subseteq E$ with $|S| = p - 1$

$$\det L(w|w) = \sum_{\substack{S \subseteq E \\ |S| = p-1}} |\det(D(w|S])|^2$$

To prove the Matrix-Tree Theorem it suffices to show the proposition on the left (proof of that later).

Proof of Matrix-Tree Theorem

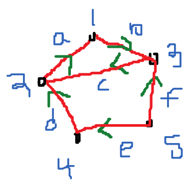
$\det(w|S] = \pm 1$ iff (V, S) is a spanning tree of G (by the Proposition)
Otherwise, $\det D(w|S] = 0$. Hence

$$\det L(w|w) = \sum_S \det D(w|S] \times \det D^T[S|w) = \sum_S |\det D(w|S]|^2 = \kappa(G)$$

Proof of Proposition

Have $D_{(V \times E)}$. Every column has exactly one $+1$ and one -1 and the rest 0.
Delete $|R|$ rows and keep $|S|$ columns. So there are $|V| - |R| = |S|$ rows and the submatrix $D(R|S]$ is square.

Consider the graph (V, S). Suppose it contains a cycle C. Consider the columns of D corresponding to edges in the set C. This set of columns is linearly dependent.



$$D = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$e + d - c - f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 0$$

Sum the columns in C with ± 1 signs according to whether e agrees in direction with the orientation around C.

Missing Lectures, Extra Content

December-05-11 1:35 PM

Section 1 of "Combinatorics of Electrical Networks"
Not on exam

Theorem (Euler)

A graph G has a trail T passing through every edge exactly once iff G has at most 2 vertices of odd degree.
(An Euler tour)

Plane Graph Numerology

Give examples of connected plane graphs with the following properties:

- 3-regular
- Every face has degree 4 or 7

Use handshake for faces and Euler's formula