Intro to Graph Theory

September-09-13 11:29 AM

Note
Notes on blackboard are frequently missing words/parts of sentences or have other "typos" so if you can't make sense of some notes it's probably because it actually doesn't make sense as written.

Graph Theory

- Matchings
- Connectivity
- Planarity

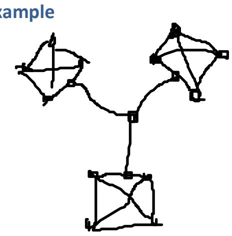
Concepts

Matchings

Does a perfect matching exist?

Simple proof that one exists: find one.
How to prove that no perfect matching exists?
If odd number of vertices then definitely not.

Example



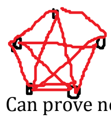
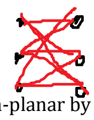
No perfect matching
$S \subseteq V(G)$
number of components of $G \backslash S$ with an odd number of vertices is larger than S

Connectivity

Menger's Theorem

Planarity

Can prove a graph is planar by drawing it on a plane.
How to prove a graph is not planar?
Non-planar examples:
$K_5, K_{3,3}$

Can prove non-planar by finding subgraph that is a subdivision of K_5 or $K_{3,3}$

Colouring

Not easy to certify that a graph is not 3-colourable.
Easy to certify not 2-colourable: No cycles.

Automorphisms & Products

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Isomorphism

An isomorphism from a graph G to a graph H is a bijection from $V(G)$ to $V(H)$ such that if $u, v \in V(G)$ and $u \sim v$ then $f(u) \sim f(v)$ and if $u \nsim v$ then $f(u) \nsim f(v)$

Notation:

$u \sim v$ means $(u, v) \in E(G)$

Graph

vertices $V(G)$
edges $E(G)$ - unordered pairs of vertices

Automorphism

An automorphism of G is an isomorphism of G to itself.

For example, with G the first 5-cycle, the bijection
0→1, 1→2, 2→3, 3→4, 4→0
is an automorphism.

Vertex Transitive

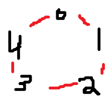
A graph is vertex-transitive if, for each pair of vertices u, v there is an automorphism of the graph that maps u to v.
(If G is vertex transitive it must be regular)

Circulants

A class of vertex transitive graphs.
We construct G as follows. Choose a positive integer n and such that $V(G) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$
Choose a subset L of $\{0, 1, \ldots, n-1\}$ such that
a) $0 \notin L$
b) if $x \in L$ then $-x \in L$
We declare $\{i, j\}$ to be an edge if $j - i \in L$

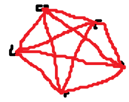
Example

$n = 5, L = \{1, -1\}$



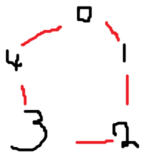
$L = \{2, -2\}$

Graph Equality

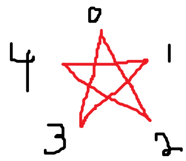


This isn't a graph. It is a picture of a graph. The vertices are not labelled.

G



H



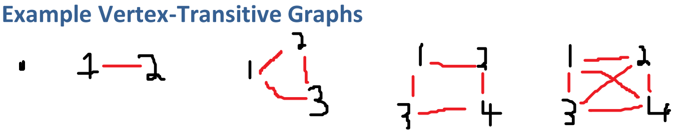
G and H are not equal, but they are isomorphic
Mapping:

G	→	H
0		0
1		2
2		4
3		1
4		3

Can verify that all edges and non-edges match up.

Note: Since $G \neq H$, the mapping given above is not an automorphism on G

Example Vertex-Transitive Graphs



Products

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Cubelike Graph

The vertices of a cubelike graph are the binary vectors of length d. (So there are 2^d vertices)
Choose a subset C of non-zero binary vectors of length d

Two vertices x and y are adjacent if their difference $y - x \in C$
A cubelike graph is regular with degree $|C|$

Statement

If G is cubelike (on 2^d vertices) and a is a binary vector of length d, the bijection $T_a: V(G) \to V(G)$ given by
$T_a(x) = x + a = x + a$
is an automorphism of G

Remark

If $a = y - x$ then $T_a(x) = x + y - x = y$. So there is an automorphism that maps x to y.

Products

Cartesian Product

Suppose G and H are graphs. The **Cartesian product** $G \square H$ is defined as follows.
Two pairs (a_1, b_1) and (a_2, b_2) are adjacent if

a) $a_1 = a_2$ and $b_1 \sim b_2$
b) $a_1 \sim a_2$ and $b_1 = b_2$

Remark

$G \square H$ and $H \square G$ are not equal in general (because $V(G) \times V(H) \neq V(H) \times V(G)$), but these graphs are isomorphic, the map that sends (a, b) to (b, a) is an isomorphism.

Remark

The d-cube is the Cartesian product of d copies of K_2

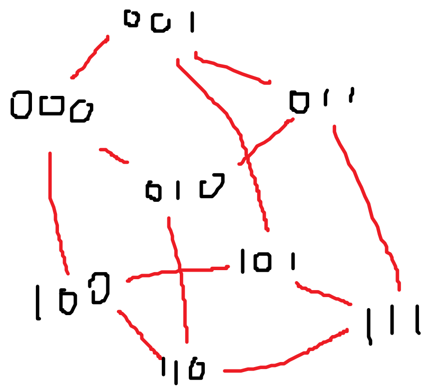
Direct Product

$G \times H$
The vertex set is $V(G) \times V(H)$ and (a_1, b_1) is adjacent to (a_2, b_2) if $a_1 \sim a_2$ and $b_1 \sim b_2$

Example

$V = \{00, 01, 10, 11\}$
$C = \{01, 10\}$

If $d = 3, C = \{001, 010, 100\}$ we get



Proof of Statement

If $x, y \in V(G)$, then
$T_a(x) - T_a(y) = (x + a) - (y + a) = x - y$
Hence $T_a(x) - T_a(y) \in C$ iff $x - y \in C$
Therefore $T_a(x) \sim T_a(y)$ iff $x \sim y$. ■

Example Cartesian Product

$G = 1 - 2, \quad H = a - b - c$

$$\begin{array}{ccccc} (1,a) & - & (1,b) & - & (1,c) \\ | & & | & & | \\ (2,a) & - & (2,b) & - & (2,c) \end{array}$$

Example Direct Product

$G = H = P_3 = 1 - 2 - 3$

11	12	13
21	22	23
31	32	33

Connectivity

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Cut-vertex

A cut-vertex in a graph G is a vertex u such that $G\backslash u$ has more connected components than G does.

Subgraph

$G, V(G), E(G)$

H **subgraph** if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

Special cases:

Induced subgraph: Pick vertex set, all possible edges between those vertices

Spanning subgraph: $V(H) = V(G)$

Component

We could define a component of the graph G to be subgraph H such that H is connect and if H' is a subgraph of G that properly contains H, then H' is not connected.

We can say H is a maximal subgraph of G that is connected. Where maximal is defined via inclusion.

H is inclusion-maximal.

2-Connected

A graph G is 2-connected if each pair of distinct vertices ab are joined by two distinct paths, P and Q so that $V(P) \cap V(Q) = \{u, v\}$ (We say P and Q are internally disjoint)

Block

A graph is a block if it is connected and does not have a cut vertex.

A subgraph H of G is a block if it is maximal (by inclusion) subject to not having a cut vertex and being connected.

Lemma

Suppose e, f, g are edges in a connected graph G. If e and f lie on a cycle, and f and g lie on a cycle, then e and g lie on a cycle.

Proof in Notes

One consequence of this is that "is equal to or lies on a cycle with" is an equivalence relation on the edges of G. The equivalence classes are the blocks of G (but we haven't proved this yet).

Theorem

For a connected graph G on at least three vertices, the following statements are equivalent:

a) G is 2-connected
b) Any two edges lie on a cycle
c) Any two vertices lie on a cycle

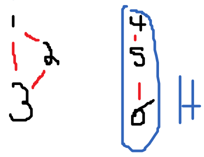
Certify connected by providing paths from each node to a common node

Certify not connected by partitioning the vertices into 2 sets such that there are no edges between them.

Any non-leaf vertex in a tree is a cut vertex.

A cycle does not have a cut vertex.

Example



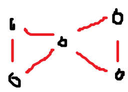
H is a component.

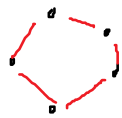
Notes on 2-Connectivity

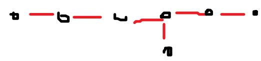
Any cycle is 2-connected. No tree is 2-connected because any 2-connected graph contains a cycle.

A 2-connected graph cannot contain a cut vertex.

Example Blocks







A block is an induced subgraph.

(1) Blocks are the triangles
(2) Only block is the whole graph
(3) Blocks are the edges (with endpoint vertices)

Proof of Theorem

a $\Longleftrightarrow$ Since 2 internally disjoint paths between 2 vertices form a cycle.

b $\Rightarrow$ c Given 2 vertices, pick two distinct edges (distinct possible because ≥ 3 vertices) such that each edge is adjacent to one of the vertices. Then there is a cycle connecting the edges $\Rightarrow$ connecting the vertices.

a $\Rightarrow$ b?

We prove that if G is 2-connected then any two distinct edges lie on a cycle.

Let e and f be distinct edges with the vertex v in common. (Other vertices u and w, respectively).

Since G is 2-connected, $G\backslash v$ is connected. So there is a path P in $G\backslash v$ joining u and w.

This path together with the path $u \rightarrow w$ is a cycle that contains e and f.

Now we prove the claim by induction on the distance between e and f.

Suppose e and f are joined by a path of length k

Let h be the edges on this path overlapping f.

Assume that k is minimal. By induction on k, we have that e and h lie on a cycle. By the first part of the proof, h and f lie on a cycle.

By the lemma, it follows that e and f lie on a cycle. ■

2-Connectivity

September-16-13 12:00 PM

2-Sum

Suppose G and H are graphs where $V(G) \cap V(H) = \{u, v\}$
and $\{u, v\}$ is an edge in G and in H.
The 2-sum of G and H is the vertex set $V(G) \cup V(H)$
The edge set is the symmetric difference $E(G) \oplus E(H)$ of $E(G)$ and $E(H)$

Theorem

If H is the 2-sum of two 2-connected graphs, then H is 2-connected.

Corollary

If G & H are 2-connected and $|V(G) \cap V(H)| = 2$ then $G \cup H$ is 2-connected.

Lemma

If G is a connected vertex-transitive graph on at least three vertices it is 2-connected.

Proof of Theorem

Suppose H is the 2-sum of G_1 and G_2. (using vertices u and v). Let e denote the edges uv.
Suppose a and b are distinct vertices in G_1. Since G_1 is 2-connected, there is a cycle in G_1 that contains a & b. If this cycle does not contain e then a & b lie on a cycle in H.
If e is in this cycle, we note that there is cycle in G_2 containing u and v so there must be a path between u and v that does not contain e. Using this path we can construct a cycle in H through a & b.
Similarly, any two vertices in G_2 lie in a cycle in H.

So suppose $a \in G_1$ and $b \in G_2$

Proof of Lemma

The proof depends on the claim that if u is a cut vertex and ψ is an automorphism then $\psi(u)$ is a cut vertex. To prove the claim, we show that $G \setminus u \cong G \setminus \psi(u)$
For this we just need to check that the restriction of ψ to $V(G)\setminus$u is an isomorphism from $G \setminus u$ to $G \setminus \psi(u)$. This restriction is a bijection and it is an isomorphism because ψ is.
This shows that if G is vertex transitive and one vertex is a cut vertex, all vertices are cut vertices.
But if G is connected, it has a spanning tree T. If this vertex u has degree one in T then u is not a cut vertex in T. i.e. $T \setminus u$ is connected.
Therefore, $G \setminus u$ is connected and so u is not a cut vertex in G. ■

- [x] Read through section 1.4 (on blocks)

Matchings & Covers

September-18-13 12:00 PM

Matchings

A matching on a graph is a set of vertex-disjoint edges.

A matching M is maximal by inclusion if there is no matching N such that $M \subseteq N$ and $N \neq M$.

Vertex Cover

A vertex cover in a graph is a set of vertices such that each edges of the graph contains at least one vertex in set.

König's Theorem

In a bipartite graph, the max size of a matching equals the minimum size of a vertex cover.

Lemma (3.1.2)

Suppose u and v are vertices in a graph G and no matching of maximum size misses both of them. Suppose M_u and M_v are maximum matchings that miss u and v, respectively.
Then there is a path of even length in $M_u \oplus M_v$ that joins u to v.

Corollary

In a bipartite graph, the set of avoidable vertices is an independent set.

Avoidable

There exists a maximum matching that does not cover it

Independent

No two vertices are adjacent

Proof

- [ ] Exercise

Definition

$\nu(g)$ is the maximum size of a matching in G

Lemma (3.1.4)

If G is a graph and $A \subseteq V(G)$ then
$\nu(G) \leq \nu(G \setminus A) + |A|$ $(*)$
If equality holds, every maximum matching of G pairs vertices of A with vertices of $G\setminus$A

Claim

If A is maximal (by inclusion) such that equality holds in $(*)$ then each vertex in $G \setminus A$ is missed by a matching of $G \setminus A$ with maximum size.

Example Maximal Matching by Inclusion

$1-2-3-4$
$\{23\}$ is maximal by inclusion

If M and N are matchings, each component of the subgraph with edge set $M \cup N$ has degree ≤ 2
Since the max degree of a vertex in the graph is at most two, its components are paths & cycles.

Proof of Lemma (3.1.2)

The subgraph with edge set $M_u \oplus M_v$ is a disjoint union of paths and even cycles. Since u is not covered by M_u and v is not covered by M_v, both u and v are the ends of paths.
Note that if some path in $M_u \oplus M_v$ has odd length then it is an augmenting path for $M_u \oplus M_v$
Suppose that there is a path in $M_u \oplus M_v$ that covers u but not v. Denote it by P. Then P has even length & the edge of P on u lies in M_v.
Then $M_v \oplus E(P)$ $(E(P) =$ the edges of $P)$ is a matching of maximum size that misses u and v. Since there is no such matching, u and v must like on the ends of a path in $M_u \oplus M_v$. Since this path has even length, we're done. ■

Outline

- $M_u \oplus M_v$ consists of even paths & even cycles.
- u and v lie on the ends of paths.
- If the path P ion u does not cover v, we can construct a matching of maximum size that misses u and v.

Proof of Lemma (3.1.4)

Let M be a maximum matching and let $N = M \cap E(G\setminus$A) - the edges of M lying in $E(G \setminus A)$ (N is a matching)
Then $|M| = |N| + |M \setminus N|$
where $|M| = \nu(G)$ and $|N| \leq \nu(G \setminus A)$, each edge in $M \setminus N$ contains at least one vertex in A, so $|M \setminus N| \leq |A|$
Therefore, $\nu(G) \leq \nu(G \setminus A) + |A|$

Proof of Claim

Now suppose that A is maximal such that equality holds. Then $\nu(G) = \nu(G \setminus A) + |A|$

Such an A does exist.

> Note that $(*)$ holds with equality for $A = \emptyset$. If u is a vertex in G that lies in every maximum matching, we may take $A = \{u\}$

Assume by way of contradiction that there is a vertex u in $G \setminus A$ that lies in each maximum matching of $G \setminus A$. I claim that the set $A \cup \{u\}$ satisfies $(*)$ with equality. Since u lies in each maximum matching of $G \setminus A$, we have

> $\nu(G \setminus \{A, u\}) = \nu(G \setminus A) - 1$
> $|\{A, u\}| = |A| + 1$
> $\nu(G \setminus \{A, u\}) + |\{A, u\}| = \nu(G \setminus A) + |A| = \nu(G)$

This is impossible because we chose A to be maximal. Hence there is no vertex in $G \setminus A$ that lies in every maximum matching of $G \setminus A$. ■

Proof of König's Theorem

- [ ] Exercise

Outline: Choose A maximal so that equality holds in $(*)$. Then $G \setminus A$ is an independent set so A is a cover.

Factor-Critical Graph

September-23-13 12:03 PM

Avoidable

A vertex in a graph is **avoidable** if there is a maximum matching that does not cover it. A vertex that is not avoidable is covered

Factor-Critical Graphs

A graph is factor-critical if it is connected and each vertex is avoidable.

Lemma 3.2.1

Let u, x, v be distinct avoidable vertices in G. Then if no maximum matching misses u and x, and no maximum matching misses x and v, then no maximum matching misses u and v.

It follows that if there is a path in G from u to v, no max matching misses both u and v.

Proof

- [ ] Exercise (use induction the length of the path)

So if G is connected and each vertex is avoidable no maximum matching misses two vertices.
Thus if G factor critical, a max matching covers all but one vertex.
(Thus $|V(G)|$ is odd)

For a graph G we use $\text{odd}(G)$ to denote the number of odd components of G.
(A component of C is odd if $|V(C)|$ is odd).
If M is a perfect matching and C is an odd component of $G \setminus S$, then some edge of M contains a vertex in C and a vertex in S.
(M pairs a vertex of C with a vertex of S).

Exercise

- [ ] If $\text{odd}(G \setminus S) > |S|$, then G not have a perfect matching.

Tutte's Theorem

A graph G has no perfect matching if and only if there is a subset S of $V(G)$ such that $\text{odd}(G \setminus S) > |S|$.

Tutte-Berge Theorem

For any graph G, the number of vertices missed by a maximum matching is $\max_{S \subseteq V(G)} \{\text{odd}(G \setminus S) - |S|\}$

Corollary

A cubic graph with at most 2 cut-edges has a perfect matching.
(Cubic = 3-regular)

Proof of Lemma 3.2.1

Let M_x be a maximum matching that misses x.
Suppose N is a maximum matching that misses u and v.

u and v are covered by M_x and x is covered by N.
Consider the subgraph formed by the edges of $M_x \cup N$. One component of this subgraph is a ux-path (by Lemma 3.1.2). By the same argument, on component of $M_x \cup N$ must be an xv-path. Since both paths contain the edge of N on x, they must be equal. Hence $u = v$.
Since $u \neq v$, the matching N does not exist. ∎

Notes about Factor Critical Graphs

$\nu(G) \leq \nu(G \setminus A) + |A|$
If A is chosen maximal so that equality holds then each vertex in $G \setminus A$ is avoidable. So each component of $G \setminus A$ is factor critical.
If G is vertex transitive and connected then either G has a perfect matching or G is factor critical.

A factor-critical bipartite graph must be K_1.
The cycle C_5 is factor-critical. Any connected vertex-transitive graph on an odd number of vertices is factor-critical.

Proof of Tutte-Berge Theorem

First, each matching in G misses at least $\text{odd}(G \setminus S) - |S|$ vertices. So we have $|V(G)| - 2\nu(G) \geq \text{odd}(G \setminus S) - |S|$ for any subset S of $V(g)$.
We have to show that equality holds.
If each vertex in G is avoidable, each component in G is factor critical and we may take $S = \emptyset$.
Otherwise, choose S maximal such that $\nu(G) = \nu(G \setminus S) + |S|$
Then $n - 2\nu(G) = n - 2\nu(G \setminus S) - 2|S| = \big(n - |S| - 2\nu(G \setminus S)\big) - |S|$
Here each component of $G \setminus S$ is factor-critical and so $\text{odd}(G \setminus S) = n - |S| - 2\nu(G \setminus S)$
∎

Proof of Corollary

Assume G is a cubic with at most two cute edges. Assume by contradiction that G does not have a perfect matching.
Choose a subset S of $V(G)$ such that $\text{odd}(G \setminus S)$ is maximal (So $\text{odd}(G \setminus S) \geq 2$).

Claim:

Each odd component of $G \setminus S$ is joined to S by an odd number of edges - there are an odd number of vertices of even degree in each such component and each vertex of even degree is joined to S by an odd number of edges.
If a component of $G \setminus S$ is joined to S by exactly one edge, this edge is a cut edge.

It follows that $3(\text{odd}(G \setminus S) - 2) + 2$ a lower bound on the number of edges joining S to a component of $G \setminus S$. But $\text{odd}(G \setminus S) - 2 \geq |S|$
It follows that on average each vertex of S is jointed to at least $\frac{3|S|+2}{|S|} > 3$ vertices in $G \setminus S$. This is impossible, so we cannot assume that G has no perfect matching.

Cut Vertex

September-27-13 11:32 AM

Cut Vertex

A cut vertex in G is a vertex u such that $G \setminus u$ has more components than G. A block is a connected graph with no cut vertex. Any block is either K_2, or it is 2-connected.

The block-cut vertex graph of G is defined as follows. Its vertices are the cut vertices and the blocks of G. There is an edge joining a cut vertex to a block fi the block contains the cut vertex (and there are no other edges).
The block-cut vertex graph is bipartite, by definition. In fact, it is a tree if the graph is connected. (Exercise.)

Contractions & Deletions

September-30-13 11:38 AM

Deletion

Delete edges.
Any graph we produce from G by deleting edges is a subgraph of G

Contraction

Combine vertices (delete edge between).
In general this produces loops & multiple edges.
Contracting edge e in graph G, denote G/e

Minor

~~We say H is a minor of G if it is isomorphic to something produced by deleting & contracting.~~
A minor is any graph we get from G by

- deleting vertices
- deleting edges
- contracting edges

Subdivision

Subdividing an edges means to add a vertex in that edge.
(uv becomes ux, xv were x is a new vertex)

H is a subdivision of G if H can be produced from G by subdividing edges.

Lemma

If G is the 2-sum of H_1 and H_2 and H_2 is 2-connected, then $H_1$is a minor of G.

Theorem

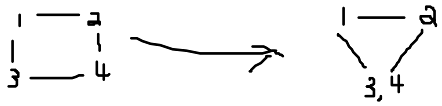
Let G be a 2-connected graph with at least three vertices. If $e \in E(G)$ then either $G \setminus e$ of G/e is 2-connected.

Multiple edges & loops - if we want to permit these, we should define a graph to consist of

a) a set of vertices
b) a set of edges
c) a relation of $V \cup E$ such that each edge is related to 1 or 2 vertices.

An edges that is related to just one vertex is a **loop**.
If G is a graph with no loops and no multiple edges, we say it is a **simple graph**.
In general, we use $\text{sim}(G)$ to denote the graph we get from G by deleting all loops and replacing all multiple edges by single edges.

Example Minor



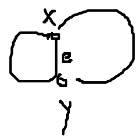
A minor is not necessarily a subgraph.
K_5 is a minor of the Petersen graph but not a subgraph.

Proof of Lemma

Starting with G, contract all the vertices of H_2. Since H_2 is 2-connected, there is a uv-path in $H_2 \setminus \{u, v\}$ so this leaves us with $\{uv\} \cup H_1 \setminus \{u, v\} = H_1$

Proof of Theorem

Suppose $e \in E(G)$ and G/e is not 2-connected. Assume $e = xy$.
Claim: The vertex produced by contracting e is a cut vertex.
Claim: G looks like



$G \setminus e$ is 2-connected?

See notes (check each pair of vertices in 4 cases)

Building 2-Connected Graphs

October-02-13 11:43 AM

Lemma (1.6.1)

Assume G is the 2-sum of 2-connected graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \{x, y\}$. If C is a cycle in G that passes through x and y, and uses vertices in $V(G_1) \setminus V(G_2)$ and in $V(G_2) \setminus V(G_1)$, then C is the 2-sum of a cycle in G_1 and a cycle in G_2

Lemma (1.6.2)

Suppose G is 2-connected and $\{x, y\}$ is a cutset in G. Then there are 2-connected graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{x, y\}$ and

a) if $x \nsim y$ then G is the 2-sum of G_1 and G_2
b) if $x \sim y$ then $G \setminus \{xy\}$ is the 2-sum of G_1 and G_2

Theorem

If G is 2-connected, one of the following holds

a) G is a cycle
b) there is an edge e such that $G \setminus e$ is 2-connected.
c) G is the 2-sum of a 2-connected graph and a cycle.

Proof of Lemma (1.6.1)

- [ ] Read notes

Proof of Lemma (1.6.2)

Assume $x \nsim y$. Then G is the 2-sum of graphs G_1 and G_2. We must show that G_1 and G_2 are 2-connected.

- [ ] Exercise: prove that any two vertices in G_1 lie in a cycle (See notes)

Otherwise, $x \sim y$. If $e = xy$ then G/e is not 2-connected (the image in G/e of e is a cut vertex and so $G \setminus e$ must be 2-connected.

■

Proof of Theorem

Call a subgraph H of G a **good** subgraph if it is a cycle, a 2-connected subgraph of G plus an edge, or a 2-sum of a 2-connected graph and a cycle.
Since G is 2-connected, it contains cycles and so it has good subgraphs.

If H is a spanning subgraph of G and H is good then G is good. (Why? Can keep adding edges according to rule (b))
If there is no good spanning subgraph choose a good subgraph H with as many vertices as possible. Since G is connected there is a vertex u in H adjacent to vertex v not in H. Since H is 2-connected, u has a neighbour w in H. Since G is 2-connected, there is a cycle containing the edges uv and vw.

Directed Graphs

October-04-13 11:33 AM

Directed Graph
Consists of a vertex set and an arc set of ordered pairs of vertices.

Oriented Graph
Directed graphs where any two arcs are joined by at most one arc.

Underlying Graph
Each directed graph has an underlying graph which has the same vertex set and has $u \sim v$ if and only if uv or vu is an arc in the directed graph.

Connectivity
A directed graph is **weakly connected** if its underlying graph is connected. A **weak component** is a component of the underlying graph.

A directed graph is **strongly connected** if, for each pair of vertices u, v, there is a path on the directed graph from u to v.

A **strong component** S of a directed graph D is a sub-directed graph of D which is maximal subject to being stringly connected.

~~Lemma~~ False
A directed graph is strongly connected if and only if each pair of vertices lies on a cycle.
False. Counterexample: 

Lemma
If S and T are distinct strong components of a directed graph and there is an arc from a vertex $u \in S$ to a vertex $v \in T$, then there is no xy where $x \in T$ and xy where $x \in T$ and $y \in S$

Theorem
Let D be a directed graph whose strong components are $S_1, \dots, S_m$. Let Q be the directed graph with $S_1, \dots, S_m$ as vertices, where (S_1, S_g) is an arc if $i \neq j$ and there are vertices $u \in S_i$ and $v \in S_j$ such that uv is an arc of D. Then Q is acyclic - it contains no cycle.

Source and Sink
A vertex in a directed graph is a sink if its outdegree is zero.
It is a source if its indegree is zero.

Lemma
If D is a weakly connected directed graph and for each vertex u, the indegree and outdegree are equal, then D is strongly connected.

Orientation
If D is a directed graph constrained by assigning an orientation to each edge of G (G undirected), we say D is an orientation.

Lemma
If G is 2-connected, it has a strongly connected orintation.

Proof
Exercise

Remark
A connected graph has a strongly connected orientation if and only if it does not have a cut edge.

Proof of Lemma
Assume by way of contradiction that D is not strongly connected, and let $C_1, \dots, C_m$ be its strong components. Let D' be the directed graph with $C_1, \dots, C_m$ as vertices, as defined before.

Exercise
Any acyclic directed graph has both sinks and sources.
Idea: Take longest path. Not a cycle so first vertex is source, last is sink.

We can assume without loss that C_m is a sink.
Note that in any directed graph, the sum of the indegrees is equal to the sum of the outdegrees.
We also note that since C_m is a sink, if $u \in V(C_m)$ then $\text{outdeg}_{C_m}(u) = \text{outdeg}_D(u)$
(Because if $u \in V(C_m)$ and $w \notin V(C_m)$ then vw is not an arc). We have

$$\sum_{u \in V(C_m)} \text{indeg}_{C_m}(u) = \sum_{u \in V(C_m)} \text{outdeg}_{C_m}(u) = \sum_{u \in V(C_m)} \text{outdeg}_D(u)$$

$$= \sum_{u \in V(C_m)} \text{indeg}_D(u)$$

But since there are vertices in C_m dominated by vertices not in C_m, the last sum is greater than

$$\sum_{u \in V(C_m)} \text{indeg}(u)$$

So we have a contradiction, and we conclude that D is strongly connected. ■

Example
Directed Circulants

Choose an integer m and a subset L of $\mathbb{Z}_m \setminus 0$. Define a digraph with vertex set $\mathbb{Z}_m$ where uv is an arc if and only if $v - u \in L$
The map T_a that sends u to $u + a$ $(a \in \mathbb{Z}_m)$ is an automorphism. It follows that all in-degrees are equal and all out-degrees are equal. So for each vertex, the in-degree is equal to the out-degree. Hence if a directed circulant is weakly connected, it is strongly connected.

Menger's Theorem

October-09-13 11:42 AM

Menger's Theorem

Let D be a directed graph and let s, t be distinct vertices in D.

The maximum number of arc-disjoint s, t paths equals the minimum size of a set of arcs C such that there is no st-path in $D \backslash C$.

Submodular Functions

Let Ω be a set and let f be a real-valued function on the subsets of Ω. We say f is submodular if, for each pair of subsets S & T of Ω,

$f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$

Note

Menger's Theorem implies that if $|V(G)| \geq k + 1$ and there is no vertex cutset of size less than k, then any two vertices are joined by k internally disjoint paths.

Example Submodular Functions

Example 1

$f(S) = |S|$

Example 2

Let D be a directed graph. If $S \subseteq V(D)$, let $f(S)$ be the number of arcs in uv such that $u \notin S$ and $v \in S$. (So $f(S)$ counts incomming arcs).

Claim: f is submodular.

$f(S) + f(T) = f(S \cup T) + f(S \cup T) + \#\{\text{arcs between } S \text{ and } T\}$

- [ ] Proof: Exercise

Let Ω be the vertex set of the graph G. If $S \subseteq \Omega$, let $f(S)$ count the edges uv with $u \notin S, v \in S$.

Example 3

Let G be the biparition with colour classes A and B. If $S \subseteq A$, define

$f(S) = |N(A)| =$ vertices adjacent to something in A

Proof of Menger's Theorem

The proof goes by induction on the number of arcs in the directed graph. Consider subsets that contain t but not s. Let k be the minimum value of $f(T)$ for such subsets, where f is defined as in example 2.

We have to prove that there are k arc-disjoint paths from s to t.

Call a subset T tight if $f(T) = k$

Claim

If S and T are tight, so are $S \cup T$ and $S \cap T$.

Proof

If S and T are tight,

$2k = f(S) + f(T) \geq f(S \cap T) + f(S \cap T)$

$t \in S \cap T$ so $S \cap T \neq \emptyset$

$f(S \cap T), f(S \cup T) \geq k \Rightarrow f(S \cap T) = f(S \cup T) = k$

Suppose xy is an arc in D that is not incomming to some tight subset (not incomming to any). Then when we delete xy we still have $f(T) = k$ for each tight subset of $D \setminus xy$. So Menger's theorem holds in $D \setminus xy$ (by structural induction). This gives us k arc-disjoint st-paths in $D \setminus xy$ and these are arc-disjoint paths in D.

Claim

There is an arc su, where $u \neq t$ and su is is incoming for some tight subset.

Proof

- [ ] Otherwise, there is only one arc on s and it is st and $k = 1$ and we are done.

Let T be the intersction of all tight subsets for which su is incoming. Note that T is tight.

3-Connected Graphs

October-16-13 11:47 AM

Theorem

Let G be a graph with at least five vertices. If G is 3-connected, there is an edge e such that $\text{sim}(G\ /\ e)$ is 3-connected.

Proof of Theorem

Assume G is 3-connected. $e = \{x, y\} \in f(G)$. Suppose that $\{u, v\}$ is a cutset in $G\ /\ e$
? If $\{u, v\} \cap \{x, y\} = \emptyset$, then $\{u, v\}$ is a cutset in G, which is impossible. So there is a vertex z in G such g and the contraction of e is a cutset in $G/\ e$. Then $\{x, y, z\}$ is a cutset in G.

So if $G/\ e$ is not 3-connected, there is vertex z which the vertices of e forms a cutset of G of size three. Choose e so that the largest component of $G\backslash$cutset is as large as possible. Let H be this component.
Since z is a cutvertex in $G \setminus \{x, y\}$, it follows that g has a neighbour not in $H \cup \{x, y\}$. Denote this neighbour by u and let f be the edge zu. Suppoze G/f is not 3-connected. Then there is a cut vertex v in $G\backslash \{z, u\}$. Se show that $v \notin H$.

Since G/z is 2-connected and $\{x, y\}$ is a vertex cutset in G/z, by Lemma 1.6.2 it follows that $H \cup \{x, y\}$ is 2-connected. Since neighter z nor u are vertices of N and v is a cut vertex in $G/\ \{u, z\}$, it follows that $v \notin V(H)$
If $v \notin V(H)$, then $(H \cup \{x, y\})/\ v$ is connected graph with more vertices than H, and it is contained in a component of $G/\ \{z, u, v\}$. This a contradiction to the assumption that for e in $E(G)$, the contraction $G/$ is not 3-connected. ■

Spanning Trees and Laplacians

October-18-13 11:32 AM

Graphs have parallel edges, no loops.

Spanning Tree

A spanning tree of a graph G is a subgraph T of G such that

- T is a tree (connected, acyclic)
- $V(T) = V(G)$

Notation

$\tau(G)$ is the # of spanning trees of G.

For $e \in E(G)$

- $G \backslash e$ is G with e removed
- G/e is G with e contracted

Keep parallel edges, remove loops.

Lemma 1

If $e \in E(G)$ then $\tau(G) = \tau(G/e) + \tau(G \backslash e)$

Laplacian of a Graph

Given a graph G

Degree Matrix

The degree matrix $\Delta(G)$ has $V(G)$ labelling both sides.

$$\Delta(G)_{v,u} = \begin{cases} \deg(v) & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}$$

Adjacency Matrix

The adjacency matrix $A(G)$ has $V(G)$ labelling both sides.

$$A(G)_{v,u} = \begin{cases} \#\text{ edges between } v \,\&\, u & \text{if } v \neq u \\ 0 & \text{if } u = v \end{cases}$$

Laplacian Matrix

The Laplacian Matrix of G is

$Q(G) \coloneqq \Delta(G) - A(G)$

Notation

- Write Q for $Q(G)$ when G is clear from context
- If $S \subseteq V(G)$ then $Q[S]$ denotes Q with rows and columns indexed by S removed.
- M is a square matrix, $M[i|j]$ is M with i-row and j-column removed.
- $V(G) = \{1, 2, \dots, n\}$

Lemma 2

For any graph G, $\det(Q) = 0$

Lemma 3

If $e \in E(G)$ and $e = uv$ then

$\det(Q[u]) = \det(Q(G \backslash e)[u]) + \det(Q(G/e)[e_{uv}])$

e_{uv} is the vertex in G/e for the contracted uv.

Theorem

Let G be a graph with $|V(G)| \geq 2$. Then for any $u \in V(G)$,

$\tau(G) = \det(Q[u])$

Corollary

$\tau(K_n) = n^{n-2}$ for $n \geq 2$

Proof of Lemma 1

Spanning trees of G that do not contain e

⇄ (bijection)

Spanning trees of $G \setminus e$

Spanning trees of G that contain e

⇄

Spanning trees of G/e

Since each spanning tree of G either contains e or it does not,

$\tau(G) = \tau(G \setminus e) + \tau(G \setminus e)$

Proof of Lemma 2

Let $\mathbf{1}_n$ be the all one vector of length n.

Then $Q\mathbf{1}_n = \mathbf{0} \Rightarrow \det(Q) = 0$

Proof of Lemma 3

Note that $Q[u]$ and $Q(G \setminus e)[u]$ have all of the values being the same, except $Q[u]_{v,v} = Q(G \setminus e)[u]_{v,v} + 1$

Cofactor expansion

$$\det(M) = \sum_{i \text{ column index}} (-1)^{i+j} M_{ij} \det(M[i|j]), \qquad \forall j$$

$\det(Q[u]) = \det(Q(G \setminus e)[u]) + (-1)^{u+v} \times 1 \times \det(Q[u][v])$

$= \det(Q(G \setminus e)[u]) + \det(Q[u,v]) = \det(Q(G \setminus e)[u]) + \det(Q(G/e)[e_{uv}])$

■

Proof of Theorem

Let $u \in V(G)$

Suppose u is an isolated vertex

Let $C_1, C_2, \dots, C_k$ be the components of $G \setminus \{u\}$

Then

$$Q[u] = \begin{bmatrix} Q(C_1) & 0 & \cdots & 0 \\ 0 & Q(C_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & Q(C_k) \end{bmatrix}$$

Then $\det Q[u] = \prod_{i=1}^{k} \det Q(C_k) = 0$

(Lemma 2)

In general, apply induction on $|E(G)|$, $|E(G)| = 0$

$\det(Q[u]) = \det(Q(u \setminus e)[u]) + \det(Q(G/e)[e_{uv}]) = \tau(G \setminus e) + \tau(G/e) = \tau(G)$

Proof of Corollary

Let J be the $(n-1) \times (n-1)$ all-one matrix. Note that

$Q(K_n)[n] = nI_{n-1} - J$

where I_n is the size n identity matrix

$(nI_{n-1} - I)x = \lambda x \Leftrightarrow (n - \lambda)x = Jx$

So λ is an eigenvalue of $Q(K_n)[n] \Leftrightarrow n - \lambda$ is an eigenvalue of J

The eigenvalues of J: 0, 0, ..., 0, $n-1$

(n-2) copies of 0

Eigenvalues of J, $n, n, \dots, n, 1$

(n-2 copies)

$\det(Q(K_n)[u]) = n^{n-2}$

Incidence Matrices

October-21-13 11:44 AM

Incidence Matrices over $\mathbb{R}$

Let G be a graph with m vertices and n edges. Choose an orientation of G. We construct the incidence matrix of this orientation as follows. The rows correspond to the vertices and the colums correspond to edges.
The ij-entry of the indices is 1 if vertex i is the head of the jth arc, -1 if it is the tail, and otherwise 0.

Lemma 1

Let G be a graph with Laplacian matrix L. If M is the incidence matrix of an orientation of G, then $MM^T = L$

Lemma 2

If G has n vertices and c components then $\text{rank}(M) = n - c$

Corollary 1

If G is connected then $x^T M = 0$ iff x is constant.

Exercise

The sum of the columns of M indexed by a cycle is zero if correctly oriented.
The set of columns is linearly dependent if the edges form a cycle.

Lemma 3

A set of columns of M is linearly independent if and only if the corresponding subgraph has no cycles - i.e. it's a forrest.

Proof of Lemma 1

The entries of MM^T are the inner product of the rows of M. The inner product of a row with itself is the degree of the associated vertex. The inner product of rows i and j, $i \neq j$, is zero if $i \nsim j$ and is $-1 \times 1 = -1$ if $i \sim j$

Linear Algebra

If $T: U \to V$ is a linear transformation then
$\dim\big(\text{range}(T)\big) + \dim(\ker(T)) = \dim\big(\text{domain}(T)\big)$

Proof of Lemma 2

We think of M as mapping in vector a^T to $a^T M$
So the domain is $\mathbb{R}^n$ and the image is the row space. We can compute dim(row space) if we have the dimension of the kernel.

If b is a column of M with $b_i \neq 1$ and $b_j = -1$ then $a^T b = 0$ iff $a_i = a_j$. So $a_i = a_j$ for every pair of vertices i, j such that $i \sim j$. We see that $a^T M = 0$ iff a has the same value for all of the vertices in each component.

The vectors constant on components form a vector space of dimension c.
hence $\dim(\ker(T)) = c$.
$\text{rank}(T) = \dim\big(\text{domain}(T)\big) - \dim\big(\ker(T)\big) = n - c$

Note

If M is the incidence matrix of an orientation of G, then
$rk(M) = |V(G)| - \#$ of components of G

In constructing the incidence matrix, multiple edges cause no problems - we just get a number of copies of the same column. If N is a submatrix of M, formed by choosing some of its columns, then N is the incidence matrix for a spanning subgraph of G. If the subgraph is S then
$\text{rank}(N) = |V(S)| - |\text{components}(S)| = n - |\text{components}(S)|$

Proof of Lemma 3

If f is a subgraph of G and contains a cycle C, then the columns of M indexed by the edges of C are linearly dependent. So if a set of columns is linearly independent, the subgraph has no cycles.
We have to show that if S has no cycles, then the columns are linearly independent.
If S is a forest, $|E(S)| = n - |\text{components}(S)| = \text{rank}(N)$
N is the incidence matrix of S
If $\text{rank}(N)$ is equal to the number of columsn of N then the columns are linearly independent.

Planarity

October-23-13 12:10 PM

Barycentric Embedding

If $v_1, \dots, v_k$ are vertices in $\mathbb{R}^n$, their **barycentre** is $\frac{1}{k}(v_1 + \cdots + v_k)$

If G is a graph and S is a subset of vertices of G, we can construct an embedding of G in $\mathbb{R}^n$ as follows:

a) Assign vectors $v_1, \dots, v_s$ to each vertex in S (where $s = |S|$)
b) If $u \in V(G) \setminus S$ then u must lie at the barycentre of its neighbours

Convex Sets in $\mathbb{R}^d$

A subset C of $\mathbb{R}^d$ is confex if, for each pair of points $x, y \in C$, all points on the line segment joining x to y belong to C.

If x and y are vectors in $\mathbb{R}^d$ then the line segment that joins them consist of all vectors $ax + (1-\lambda)y,\ 0 \le \lambda \le 1$

Convex Combination

We say y is a convex combination of $x_1, \dots, x_m$ if $y = a_1 x_1 + \cdots + a_m x_m$ where $a_i \ge 0$ and $\sum_i a_i = 1$

If S and T are convex sets, so is $S \cap T$.

Convex Hull

If $S \subseteq \mathbb{R}^d$ then the intsersection of all convex sets that contain S is a convex set and is called the **convex hull** of S.

Lemma

If S is a finite set of points, then the convex hull of S consists of the convex combinations of the elements of S.

Barycentric Embedding

Suppose we have embedding of the vertices of G in $\mathbb{R}^2$. This embedding is barycentric at the vertex s if

$$x_i = \frac{1}{d_i} \sum_{j \sim i} x_j$$

Here, x_k is the image of the vertex k and d_k is its degree.

Equivalently,

$$d_i x_i = \sum_{j \sim i} x_j \text{ or } d_i x_i - \sum_{j \sim i} x_j = 0$$

The coefficients of this last equation are entries in row i of the laplacian of G.

In practice we proceed as follows. Choose a cycle C such that $G \setminus C$ is connected and embed the vertices of C as the vertices of a regular polygon in the plane. We want to find an embedding that is barycentric on the vertices not in C.

Our data structure is a $|V(x)| \times 2$ matrix

$$\begin{bmatrix} X_C \\ - \\ X_{\bar{C}} \end{bmatrix}$$

Where the rows of X_c give the embedding of the cycle

$$\begin{bmatrix} L_{00} & | & L_{01} \\ - & + & - \\ L_{01}^T & | & L_{11} \end{bmatrix} \begin{bmatrix} X_C \\ - \\ X_{\bar{C}} \end{bmatrix} = \begin{bmatrix} * \\ - \\ 0 \end{bmatrix}$$

thus our barycentric condition is equivalent to

$L_{01}^T X_C + L_{11} X_{\bar{C}} = 0$

If L_{11}^{-1} exists then $X_{\bar{C}} = L_{11}^{-1} L_{01}^T X_C$

In particular, if L_{11} is invertible, then $(*)$ has a unique solution. I claim that L_{11} is invertible if $G \setminus C$ is connected. To see this, let M be the sincidence matrix of G. Then we can write

$$M = \begin{bmatrix} M_C \\ - \\ M_{\bar{C}} \end{bmatrix}$$

Now, $MM^T = L$ and

$$\begin{bmatrix} M_C \\ - \\ M_{\bar{C}} \end{bmatrix} \begin{bmatrix} M_C^T & | & M_{\bar{C}}^T \end{bmatrix} = \begin{bmatrix} M_C M_C^T & | & M_C M_{\bar{C}}^T \\ - & + & - \\ M_{\bar{C}} M_C^T & | & M_{\bar{C}} M_{\bar{C}}^T \end{bmatrix}$$

We have $\operatorname{rank}(M_{\bar{C}} M_{\bar{C}}^T) = \operatorname{rank}(M_{\bar{C}})$ (by linear algebra)

We assume that G is connected. We are also assuming that $G \setminus C$ is connected. Given these assumptions. we need to show that the rows of $M_{\bar{C}}$ are linearly independent.

The proof of this is an exercise.

Planar Graphs Cont.

October-28-13 12:09 PM

Planar Graphs

In this context, our graphs need not be simple. (So a graph is a set of vertices, a set of edges, and an incidence relation on vertices and edges).

A drawing of a graph in a surface divides the surface into faces. All faces must be topologically equivalent to discs. (The technical term is that the embedding is cellular)
A drawing of a connected graph in the plane with no crossings is called a **planar drawing** or a **planar map**.

Each planar map has vertices, edges, and faces. This gives us an incidence relation between edges and faces.

Lemma

Suppose G is a 2-connected graph with at least three vertices. Then in any planar drawing of G, each face is a cycle and each edge lies in eactly two faces.

If G is planar, so is any minor, and so is any subdivision.

Proof of Lemma

(Sketch - see notes for details)
Note we can construct any 2-connected graph starting from a cycle by adding edges or by subdividing edges. The proof is by induction on the number of edges.

Bridges & Cycles

November-01-13 11:38 AM

Chord and Bridge

Let G be a graph and let C be a cycle in G. A **chord** is an edge of G that joins two vertices of C but is not an edge of G. A **bridge** of C is either

a) a chord, or
b) a subgraph of G formed by a component of $G\backslash C$ together with the vertices on C adjacent to the component and edges these vertices to vertices in the component.

Feet

The vertices of a bridge that are on the cycle are called the **feet** of the bridge.

Span

If u and v are consecutive feet on the cycle, the path in the cycle joining these is a **span** of the bridge.

A bridge B_1 **avoids** a bridge B_2 if all feet of one bridge are contained in one span of the the other. Two bridges that do not avoid each other are said to **overlap**.

Remarks

i) If C is a cycle in a graph G and P is a bridge then in a plane drawing, all vertices of P lie in the same face of C.
ii) If B_1 and B_2 are overlapping bridges then they lie in different faces.

Overlap Graph

If $B_1, \ldots, B_m$ are the bridges of a cycle, the **overlap graph** of C has these bridges as vertices and two bridges are adjacent if they overlap.

Claim

If G is planar and C is a cycle, then the overlap graph relative to C is bipartite.

Peripheral Cycles

A cycle C in a graph **is peripheral** if $G\backslash C$ is connected. That is, if C has at most one bridge.

Theorem

Suppose G is a 3-connected. A cycle C in G bounds a face iff and only if it is periperal.

Proof of Theorem

First, suppose C is peripheral . Then it has exactly one bridge (G is not a cycle) and in a planar drawing this bridge lies in one face of C. The other face of C is a face of the graph.
For the next step, we claim that if no two bridges overlap then G is not 3-connected.
If two bridges of G overlap, then they must be embedded in different faces of C.
$\Rightarrow C$ is not a face of the drawing of G.

Kuratowski's Theorem

November-04-13 11:32 AM

A cycle in a graph is peripheral if $G\backslash \mathrm{V}(C)$ is connected. Equivalently, C has only one bride. We proved that a face in a planar drawing of a 3-connected graph is peripheral. Conversely, any peripheral cycle must bound a face.

Corollary
A 3 connected graph has at most one drawing in a plane, up to reversal.

Kuratowski's Theorem
Theorem
A graph G can be drawn in the plane iff it does not have a subdivision of K_5 or $K_{3,3}$ as a subgraph.

There is also a minor version: (equivalent)

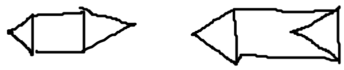
Theorem
A graph G can be drawn in the plane iff it does not have K_5 or $K_{3,3}$ as a minor.

Lemma
Let H be a cubic graph. If a graph has H as a minor, then it contains a subdivision of H

Remark
If G contains a subdivision of F, then F is a minor of G

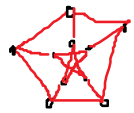
Lemma 2
If G contains K_5 as a minor, then it conatins a subdivison of K_5 or $K_{3,3}$.

Proof of Corollary
For a 3-connected graph, the faces are precisely the peripheral faces.

Example of Different Embeddings


Two different embeddings. Faces have different number of edges so not the same.

Difficulty in proving 2 versions of Kuratowski's Theorem equivalent
Petersen graph



Has a K_5 minor but does not have K_5 has a subdivision.

Proof of Lemma
The proof is by induction on $|E(G)|-|E(H)|$.
If $|E(G)|=|E(H)|$ then the result is true, because $G\cong H$
Now suppose there is an edge e such that H is a minor of $G\backslash e$. By induction, since H is cubic $G\backslash e$ contains a subdivision of H. Hence G contains a subdivion of H.
(Board says "... subdivision of e", but that doesn't make sense to me).
So we may assume that for edge e, $G\backslash e$ does not contains H as a minor. So there is an edge f such that G/f has H as a minor. If the image of f is not in the copy of H, then G/f has a subdivision of H that does not use this image, and G has a subdivision of G. So H is equal to K/f where K is a subgraph of G that contains f. By induction, K/f contains a subdivision H_1 of H and H_1 contains f.
The degree of f in H_1 is three. So the two vertices of f in K have dgree at most three in K.

Proof of Lemma 2
We proceed by induction on the number of edges. If K_5 is a contraction of a proper subgraph of G_5, then this contraction has a K_5 or $K_{3,3}$-subdivision and so, by induction, G has one too.

November-06-13 11:36 AM

Theorem

Let C be a peripheral cycle in a 3-connected graph. If G is planar, the barycentric enbedding relative to C is a planar drawing.

C is embedded as a convex polygon.

Proof of Theorem

We establish a sequence of claims

Claim 1

Let l be a line in the plane that is not parallel to any edge of the embedding and meets the embedding of C in two paints. Then ther vertices on one side of l or on l induce a connected graph.

For any vertex below the line, either it is on the cycle, or it has a vertex below it (relative to the line - since cannot have all neighbours on one side and no edges parallel to l) or it is on the same point as all its neighbours. In the 2nd case we can keep going to lower vertices until reach case 1 or 3.
So there is no difficulty unless there is a vertex which is embedded at the same point as each of its neighbours. In this case, let H be a connected component of the graph induced by the vertices maped to the same point as u. There is a vertex v in H adjacent to vertices not in H. From this we can go "uphill" or "downhill" to edges not parallel to l.
Not always, H could contains a point in C, but fine in that case.

Claim 2

There is no vertex such that u and all its neighbours lie on the same line l.

Let H be a component of the subgraph of G induced by the vertices that are on l, and have all their neighbours on l.
If all vertices of H lie on a point, then pick l such that it intersects C twice.
We can assume that H contains u. Let S be the set of neighbours of vertices in H that are not in H. Let U_1 and U_2 be the sets of vertices above and below l. Then U_1 and U_2 are connected. If we contract U_1 and U_2 and H to single vertices then the subgraph formed by those vertices and three vertices from S is K_3.

Claim 3

Let $e = ab$ be an edges that is not on C and F_1, F_2 be the two faces of C on e.
Let l be the line through the images of a and b. Then all vertices of F_1 lie on one side of l, and all vertices of F_2 lie on the other.

Claim 4

The boundary of each face is a convex polygon.
Outline of proof: each edge determines two half-planes; each face is the intersection of half-planes determined by its edges.

Claim 5

The interiors ofo the faces of disjoint.

Cuts and Flows

November-11-13 11:31 AM

We work with incidence matrices over $\mathbb{Z}_2$.
The incidence matrix of a graph G over $\mathbb{Z}_2$ has rows indexed by vertices, columns indexed by edges, and the ij-entry is 1 if the i^{th} vertex is on the j^{th} edge. Otherwise it is 0.

Lemma
$\operatorname{rank}(B) = n - c$ where $n = |V(G)|$, c is the number of components, and B is the incidence matrix of G.

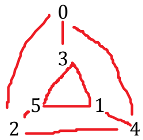
Edge Cut
If (S, T) is a partition of $V(G)$, we say that the edges that connect a vectex in S to a vertex in T form an **edge cut** (or a **cut**). The empty set is a n edge cut.

Claim
$E(G)$ is an edge cut if and only if G is bipartite.

Theorem
The edge cuts of G form a vector space of $\mathbb{Z}_2$.

Fundamental Edge Cut
If T is a spanning tree on G and $e \in E(T)$ then $T \setminus e$ has exactly two components, which partition $V(G)$. This gives us an edge cut called the fundamental edge-cut relative to T

Example Incidence Matrix
$K_2 \square K_3$



Incidence matrix
Vertices $\{0, 1, 2, 3, 4, 5\}$
Edges $\{02, 03, 04, 13, 14, 15, 24, 25, 35\}$
Incidence matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = B$$

The row space of B is a subspace of $\mathbb{Z}_2^9$. Its dimension is at most 6. ($= |V(G)|$), but the sum of the rows is 0. So its dimension is at most 5.

Remark
If U is a vector space of dimension k over $\mathbb{Z}_2$, then $|U| = 2^k$. (Look at all linear combinations of basis vectors.)
Each row of B determines a set of edges, the subset of $E(G)$ consisting of the edges that conatin a given vectex. We say each row of a **characteristic vector** (of a subset of $E(G)$).
Addition mod 2 of characteristic vectors scorrespond to symmetric difference of subsets.

Proof of Lemma
Suppose $\vec{x}B = 0$. If $e = \{i, j\} \in E(G)$, this means that $x_i + x_j = 0$ and, since we're working mod 2, we have $x_i = x_j$ if $\{i, j\} \in E(G)$
Se the "left nullspace" of B has dimension c.
Hence $\operatorname{rank}(B) = n - c$
■

If a set of edges forms a cycle in G, then the sum of the corresponding columns of B is zero.
As before (with the oriented case over $\mathbb{R}$), a set of columns is lienarly independent if and only if the edges form a forrest.

Proof of Theorem
The row space of B consists of the characteristic vectors of the edge cuts.

Claim
If x is binary $x^T B$ is the characteristic vector of an edge cut.

Even Graphs

November-13-13 11:34 AM

The row space of the incidence matrix of a graph is the cut space of the graph. What is the kernel of B? If G has exactly c components then $\operatorname{rank}(B) = |V(G)| - c$ and so
$\dim(\ker(B)) = |E(G)| - |V(G)| + c$

Suppose $Bf = 0$ and let H be the subgraph formed by the edges in the support of f.
(So f is the characteristic vector of its support.)
If $v \in V(G)$ then the v entry of Bf is the number of edges on v and in H mod 2.
Thus $Bf = 0$ if and only if each vertex in H has even degree.
So $\ker(B)$ contains all cycles.
A graph is **even** if all of its vertices have even degree. A connected even graph is also called an **Eulerian graph**.

The kernel of B can be called the flow space of G - it consists of all characteristic vectors of even subgraphs of G.
What is the subspace of $\ker(B)$ spanned by the cycles? In fact it is $\ker(B)$. To see this we have to verify the following claims:

a) If H is an even subgraph and $E(G) \neq \emptyset$ then it contains a cycle C
b) If H' is H with the edges of C deleted then the characteristic vector of $E(H')$ is the sum of the characteristic vectors of H and C.
c) H' is even.

So our claim follows by induction on $|E(H)|$.

Each chord in a spanning tree determines a unique cycle, a so-called fundamental cycle relative to the tree. For a fixed tree, the fundamental cycles are linearly independent (prove it) and so they form a basis.

$B \to$ Reduced Row Echelon Form $\to [I\ B_1]$
$[I \quad B_1] \begin{bmatrix} -B_1 \\ I \end{bmatrix} = -B_1 + B_1 = 0$
So each row of $[-B_1^T\ 1]$ is orthogonal to each row of B.

Duals

November-13-13 12:14 PM

Let G be a plane graph with dual G^*
$G^{**} = G$

1. If e is not a loop and not a cut edge then
   $(G \setminus e)^* = G^*/f$
   f is the edge in G^* that corresponds to e
2. If e is not a loop
   $(G/e)^* = G^* \setminus f$

Remark: if e is a loop then $G \setminus e = G/e$

Claim

If G is connected then G^* is connected.

Note

If the set of edges C is a cycle in G then its image in G^* is an edge cut.

We make use of the incidence matrix of G. The vectors in the row space of the incidence matrix B are the characteristic vectors of edge cuts. The vectors in $\ker(B)$ are the characteristic vectors the the even subgraphs.
If G is planar and has no cut vertex, each face is bounded by a cycle. Since each edges lies in exactly two cycles, the sum of the characteristic vectors of the faces is 0.
The matrix with these vectors as its rows is the incidence matrix of G^*. Denote this by B^*.
Each row of B has dot product zero with any row of B^* because the rows of B^* are the characteristic vectors of cycles in G. So $B^*B^T = 0$

Exercise

Let G be a planar graph with no cut vertex.
Let $f_1, \ldots, f_k$ be a st of faces of G. If the sum of the characteristic vectors of these faces is zero then $f_1, \ldots, f_k$ is the complete set of faces.

Corollary

G^* is connected

Lemma

Let G be a planar graph. Then G is bipartite if and only if G^* is even.

Proof of Lemma

If G is bipartite then the all-ones vector 1 lies in the row space of G.
Conversely, if $1 \in row(B)$ then G is bipartite. So if G is bipartite, each vector in $\ker(B)$ is the characteristic vector of an even subset.
It follows that each edge cut in G^* has even size
So each vertex in G^* has even degree.
For the converse, suppose G is even. Then $G \cdot 1 = 0$ and so $1 \in \ker(B)$.
hence $1 \in row(B^*)$ so G^* is bipartite.

Duals Cont.

November-18-13 11:31 AM

Duality

- $(G^*)^* = G$
- $(G/e)^* = G^* \setminus e$, $(G \setminus e)^* = G^*/e$ (with some caveats)
  (if e is a loop, $G/e \cong G \setminus e$)
- If S is the set of edges of a cycle in G, then its image in G^* is an edge cut.

Lemma

If G is a connected, planar graph then G^* is connected.

Lemma 2

If S is the edge set of a spanning tree in the plane graph G, then the image of $\bar{S}$ in G^* is a spanning tree.

Corollary (Euler)

IF $n = |V(G)|$ and $e = |E(G)|$ and G has a planer embedding with f faces, then $n - e + f = 2$.

Lemma 3

Two edges of a graph lie in a minimal edge cut of G if and only if they lie in a block of G.

Corollary

If G is planar and 2-connected the any dual of G is 2-connected.

Proof of Lemma

Let T be a spanning tree of G. Successively delete the chords of T from G. The dual T^* of T is a graph with one vertex, so it is connected.
Since T^* is reached from G^* by contraction, G^* must be connected.

Proof of Lemma 2

Since S contains no cycles, its image in G^* does not contain an edge cut. So if we delete the image of S from G^*, the graph left is connected. i.e. $\bar{S}$ is the edge set of a connected spanning subgraph of G.
But if $e \notin S$ then $S \cup e$ contains a cycles and therefore its image in G^* contains a cut edge. So $\bar{S} \setminus$ (image of e) is not connected. Thus the imate of $\bar{S}$ is connected but any edges is a cutedge. Hence it is a tree. ■

Proof of Corollary (Euler)

If T is a spanning tree in G then $E(G^*) \setminus E(t)$ is a spanning tree in G^*. Now $|E(T)| = n - 1$ and the number of edges in a spanning tree of G^* is $f - 1$.
So $n - 1 + f - 1 = |E(T)| + |E(G^*) \setminus E(T)| = |E(G)| = e$.
■

Proof of Lemma

If e and f are edges in the same block of G, there is a cylce C that contains them. Then $C \setminus e$ has no cycle and so it is a subset of $E(T)$ for some spanning tree T. The edges that join the two components of $T \setminus f$ form a minimal edge cut which contains e and f.

[Proof incomplete. Rest will be provided in course notes].

Proof of Corollary

By the lemma, "is equal to a lies in minimal edge cut" is an equivalence relation on the edges of G and the equivalence classes are its blocks.

If G^* has a cut vertex it has more than one block, and so G has more than one block. ■

Knots and Links

November-18-13 12:16 PM

Shadow Graph

Every link diagram has an underlying graph.
The vertices are the crossings and the paths between them are the edges.
This may have loops and multiple edges.
It is called the **shadow** of the link diagram.
This is a plane 4-regular graph

The shadow is a connected even graph, thus it is **Eulerian**.

Eulerian Walk

An **Eulerian walk** in a graph is a walk that uses each edge at most once. A closed Eulerian walk that uses every edge is an **Eulerian tour**.
Two Eulerian closed walks are equivalent if they differ only in the choice of starting vertex, or in the choice of directed.
An equivalence of closed Eulerian walks is **Eulerian cycle**.

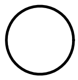
An Eulerian cycle in a (4-regular) plane graph is **straight** if we leave a vertex by the edge opposite the one we came in (relative to the embedding). In a shadow of a link each component of the link determines a straight Eulerian cycle. These cycles partition the edges of the shadow.

An **Eulerian partition** of G is a partition of the edges into Eulerian cycles.

Each Eulerian partition of a 4-regular graph determines a partition of the four edges at each vertex into two parts. There are three ways of dividing a set of size four vertices into three parts and it follows that a 4-regular graph on n vertices has 3^n Eulerian parittions.

A knot is a closed loop in 3-space. A link is a collection of disconnected knots.

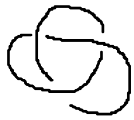
Unknot



Link Example

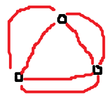
Drawings are projections on the plane of a 3d object.

Trefoil



Shadow graph

4-regular graph on 3 vertices



Question

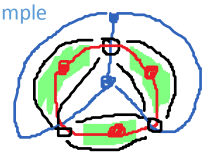
How can we determine the number of components of a link from its shadow?

Colouring

The dual of a shadow graph is bipartite. Can 2-colour the faces. For every vertex, colour the two opposite faces the same colour.

There are 2 other plane graphs that can be created: the vertices are faces of the same colour in the 2-colour (either all black or all white) and vertices edges between faces of the same colour. These graphs are duals of each other.

Example



Bicycles and Medial Graph

November-22-13 11:35 AM

Bicycles

A subset B of $E(G)$ is a **bicycle** if it is both a (edge?) cut and an even subgraph of G.
The empty set is a bicycle.
A graph with no nonempty bicycles is **pedestrian**.

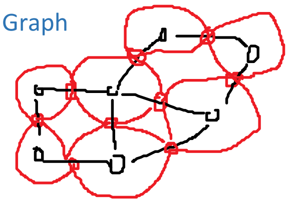
The set of bicycles of G is the intersection of the cutspace and flow space of G.

Lemma

If B is the incidence matrx of G (over $\mathbb{Z}_2$) then
$\dim(C \cap C^\perp) = n - c - \mathrm{rk}(BB^T)$

Link diagram → shadows (4-regular plane graph)
Components of the link correspond to straight Eulerian cycles in the shadow; we get a straight Eulerian partition.
Each plane Eulerian graph has a bipartite dual and so has two face graphs - one on the one faces and one on the black.

Medial Graph



Let G be a plane graph. The vertices of its medial graph are the edges of G. The edges of G are adjecent in $M(G)$ if they have a vertex in common and lie in the same face.
Note that $M(G)$ is 4-regular and plnar.
More importantly $M(G^*) = M(G)$. Equilvalently, G and G^* are the face graphs of $M(G)$.

What do the straight Eulerian cycles in $M(G)$ correspond to in G?

Bicycles

Let V be a vector space of $\mathbb{Z}_2$. If $S \subseteq V$ then
$S^\perp := \{y : \langle x, y\rangle = 0 \ \ \forall x \in S\}, \qquad \langle x, y\rangle = x^T y$
Note: $S^\perp$ is a subspace of V.

Claims

1) $S \subseteq (S^\perp)^\perp$
2) If S is a subspace of V then $\dim(S^\perp) = \dim(V) - \dim(S)$
3) Together (1) & (2) ⇒ $(S^\perp)^\perp = S$
4) If C is the cuspace of G and F is the flow space of G then $F = C^\perp$
5) So the bicycle space of G is $C \cap C^\perp$

What is the dimension of $C \cap C^\perp$

Proof of Lemma

We have a linear transformation that sends a binary vector x to Bx. We view this as a map acting on $col(B^T)$; its image consists of all vectors of the form $BB^T y$. i.e. it's $\mathrm{col}(BB^T)$
So $B: col(B^T) \to col(BB^T)$
and $\ker(B)$ is the bicycle space. By the rank + nullity theorem,
$\dim\big(\mathrm{col}(BB^T)\big) + \dim(\ker B) = \dim\big(\mathrm{col}(B^T)\big) = \mathrm{rk}(B) = n - c$
This gives the statement of the lemma. ■

Tutte Polynomial

November-25-13 11:31 AM

Tutte Polynomial

let $G = (V, E)$ be a graph. Let $c(G)$ denote the number of ocmponents of G and for each $S \subseteq E$, let $c_G(S)$ deonte the number of components of (V, S). The Tutte Polynomial $T_G(x, y)$ is defined by

$$T_G(x,y) = \sum_{S \subseteq E} (x-1)^{c_G(S)-c(G)}(y-1)^{|S|+c_G(S)-|V|}$$

Lemma

Let $G = (V, E)$ be a graph and let $e \in E$

1) If e is a loop, then $T_G(x,y) = yT_{G\setminus e}(x,y)$
2) If e is a cut-edge, then $T_G(x,y) = xT_{G\setminus e}(x,y)$
3) If e is not a loop or cut-edge, then $T_G(x,y) = T_{G\setminus e}(x,y) + T_{G/e}(x,y)$

More Properties of T_G

4) If $B_1, \dots, B_k$ are the blocks of G then
$$T_G(x,y) = \prod_{i=1}^{k} T_{B_i}(x,y)$$
5) $T_G(x,y) = T_{G^*}(y,x)$ if G is planar.

Tutte Polynomial Examples

$T_{K_1}(x,y) = (x-1)^{1-1}(y-1)^{0+1-1} = 1$
$T_{K_2}(x,y) = (x-1)^{2-1}(y-1)^{0+2-2} + (x-1)^{1-1}(y-1)^{1+1-2} = x - 1 + 1 = x$
$T_G(x,y) = (x-1)^{1-1}y(-1)^{0+1-1} + (x-1)^{1-1}(y+1)^{1+1-1} = 1 + y - 1 = y$
where G :

For bigger graphs, we would like a better way to compute T_G

Proof of Lemma

Proof of (1) and (2) are exercises.
If will be useful to view the edges sets of G/e and $G \setminus e$ as subsets of E; in particular, $E(G \setminus e) = E(G/e) = E \setminus \{e\}$
Some observations:

- Since e is not a cut-edge
  - $c(G \setminus e) = c(G)$
  - $|V(G \setminus e)| = |V|$
  - $c_{G\setminus e}(S) = c_G(S)\ \ \forall S \in E \setminus \{e\}$
- Since e is not a loop
  - $c(G/e) = c(G)$
  - $|V(G/e)| = |V| - 1$
  - $c_{G/e}(S) = c_G(S \cup \{e\})\ \ \forall\, S \in E \setminus \{e\}$

Now $T_{G\setminus e}(x,y) = \displaystyle\sum_{S \subseteq E(G\setminus e)} (x-1)^{c_{G\setminus e}(S)-c(G\setminus e)}(y-1)^{|S|+c_{G\setminus e}(S)-|V(G\setminus e)|}$

$$= \sum_{S \subseteq E\setminus e} (x-1)^{c_G(S)-c(G)}(y-1)^{|S|-c_G(S)-|V|} = \sum_{\substack{S \subseteq E \\ e \notin S}} (x-1)^{c_G(S)-c(G)}(y-1)^{|S|-c_G(S)-|V|}$$

Also, $T_{G/e}(x,y) = \displaystyle\sum_{T \subseteq E(G/e)} (x-1)^{c_{G/e}(S)-c(G/e)}(y-1)^{|S|+c_{G/e}(S)-|V(G/e)|}$

$$= \sum_{T \subseteq E\setminus e} (x-1)^{c_G(T\cup e)-c(G)}(y-1)^{|T|+c_G(T\cup e)-|V|+1}$$

$$= \sum_{T \in E\setminus e} (x-1)^{c_G(T\cup e)-c(G)}(y-1)^{|T\cup e|+c_G(T\cup e)-|V|}$$

$$= \sum_{\substack{S \subseteq E \\ e \in S}} (x-1)^{c_G(S)-c(G)}(y-1)^{|S|+c_G(S)-|V|}$$

Adding these two expressions together we get

$$T_{G\setminus e}(x,y) + T_{G/e}(x,y) = \sum_{S \subseteq E} (x-1)^{c_G(S)-c(G)}(y-1)^{|S|+c_G(S)-|V|} = T_G(x,y)$$

■

Why do we care about Tutte Polynomials?

T_G contains lots of information about G.
For a connected graph G,

? Interesting Example 1

$$T_G(1,1) = \sum_{S \subseteq G} (0)^{c_G(S)-1}(0)^{|S|+c_G(S)-|V|}$$

A term in the above sum is non-zero (± 1) iff $c_G(S) - 1 = 0$ and $|S| + c_G(S) - |V| = 0 \Leftrightarrow c_{G(S)} = 1$ and $|S| = |V| - c_G(S) = |V| - 1$
Therefore, a term is nonzero iff the corresponding subgraph is a spanning tree of $G \Rightarrow T_e(1,1)$ is the number of spanning trees of G.

Interesting Example 2

Let $p(G, k)$ denote the number of proper k-colourings of G (recall Assignment 7, Question 6). Then $p(G,x) = (-1)^{|V|-1} x T_G(1-x, 0)$

Why?

$$T_G(1-x,0) = \sum_{S \subseteq E} (-x)^{c_G(S)-1}(-1)^{|S|+c_G(S)-|V|} = \sum_{S \subseteq E} (-1)^{|S|+2c_G(S)-|V|-1} x^{c_G(S)-1}$$

$$= \sum_{S \subseteq E} (-1)^{|S|-|V|+1} x^{c_G(S)-1}$$

$$\Rightarrow (-1)^{|V|-1} x T_G(1-x,0) = \sum_{S \subseteq E} (-1)^{|S|} x^{c_G(S)}$$

☐ Exercise

Show $p(G,x) = \displaystyle\sum_{S \subseteq E} (-1)^{|S|} x^{c_G(S)}$

Interesting Example 3

Let F be the flow space of G and let C be bet the cut space of G. Then $|T_G(-1,-1)| = 2^{\dim(F \cap C)} =$ number of bicycles in G

Cuts and Flow Type

November-27-13 11:39 AM

Subspace Sum

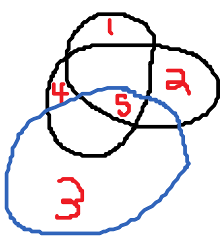
If C and F are subspaces then their **sum** is
$C + F = \{x + y : x \in C, y \in F\}$
It is the smallest subspace that contains C and F.
Also, $(C \cap F)^{\perp} = C^{\perp} + F^{\perp}$

Characterization of Edges

Suppose $e \in E(G)$ and e does not lie on a bicycle. Then if x is the characteristic vector of a bicycle, $\langle e, x\rangle = 0$
So $\langle e, x\rangle = 0$ for all x in $C \cap F = C \cap C^{\perp}$ and therefore $e \in (C \cap F)^{\perp} = C^{\perp} + F^{\perp} = F + C$
Hence $e \in C + F$, i.e. it is the sum of a cut and a flow (So e is the symmetric difference of a cut and a flow).
Suppose $e = \gamma + \phi$ where γ is a cut and ϕ is a flow. Then e lies in the cut or the flow, but not both.
If e lies in the cut then $e + \gamma = \gamma + \gamma + \phi = \phi$ and therefore $\gamma \setminus e$ is a flow. (It is a bicycle in $G \setminus e$)
On the other hand, if e lies in ϕ then $e + \phi = \gamma$ and $\phi \setminus e$ is a cut (It is a bicycle in G/e)
Suppose $e = \gamma + \phi = \gamma' + \phi'$ where γ, γ' are cuts and ϕ, ϕ' are flows. Then we have $\gamma + \gamma' = \phi + \phi'$ where $\gamma + \gamma'$ is a cut and $\phi + \phi'$ is a flow.

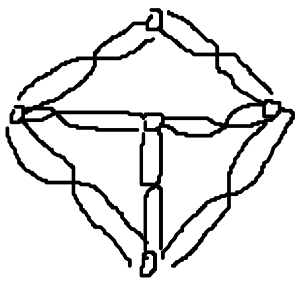
Since e does not lie in any bicycle, it follows that if $e \in \gamma$ then $e \in \gamma'$ (& e is of **cut type**).
If $e \in \phi$ then $e \in \phi'$ and we say e is of **flow type**.
Thus we have a partition of the edges of G into 3 classes - bicycle, cut, and flow.

Bicycles and Left-Right Walks



Green is left-right walk

Ribbon Graph



The **core** of a left-right walk consists of the edges it uses just once.

Lemma

The core of a left-right walk on a planar graph is bipartite.

Proof

I claim that any left-right walk on G is a left-right walk on the dual (Exercise).
Let W be a left-right walk on G. Then each vertex of G is visited an even number of times by W.
If we delete an edges visited twice by W, whats left is still even. So the core is an even subgraph. It is also an even subgraph in the dual, so it is a cut in G. Therefore, it is a bicycle. ■

Chord Diagrams & Circle Graphs

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Double Occurrence Words

Suppose we have the shadow of a knot. We can number the crossings. Then if we walk around the knot, we can list the crossings in the order we meet them. A **double occurrence word** is a sequence formed by elements of $\{1, \ldots, n\}$ where each integer occurs exactly twice.

Which double occurrence words come from the shadow of a knot?
We can convert double occurrence words to graphs known as **chord diagram**.

There is another graph whose vertices are the chords where two chords are adjacent if they intersect. This is called a **circle graph** (an **intersection graph**).

Claim

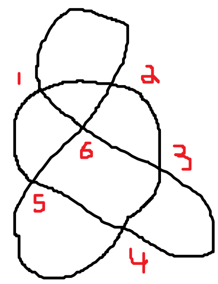
The circle graph of a knot shadow is even.

The circle graph is even if, in the double occurrence word, the number of symbols between two occurrences of the same symbol is even.

Consider the portion of the straight Eulerian tour between the two times we are at crossing i. Assume we have 2-coloured the faces of the knot shadow. As we move around the loop, the colours of the faces of the left alternates. When we return to crossing i, the face on our left is the same colour as the face on our left when we started. Hence the loop passes through an even number of crossings.

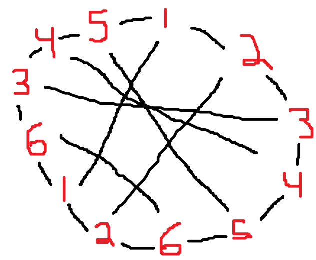
We can split each crossing in the knot shadow to produce a bent Eulerian tour. This gives a chord diagram which is planar. And a chord diagram is planar if and only if its intersection/circle graph is bipartite.

Example

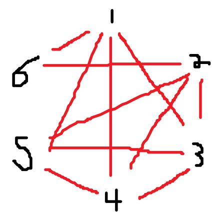


Associated double occurrence word:
1 2 3 4 5 6 2 1 6 3 4 5

Associated chord diagram



Associated circle/intersection graph



6 flips in the double occurrence word resulting in a bent Eulerian tour
1 2 3 6 2 1 6 5 4 3 4 5