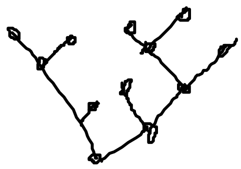
Sets

September-09-13 10:48 AM

Fully Binary Tree

Has a root and every node has either 2 children (left and right) or 0 children.



Constructions with Sets

Cartesian Product

Given two sets S, T the Cartesian product
$S \times T = \{(s,t) | \ s \in S, t \in T\}$
More generally,
$S_1 \times S_2 \times \cdots \times S_k = \{(s_1, s_2, \ldots, s_k) \mid s_i \in S_i, \quad i = 1, \ldots, k\}$

Power Set

If S is a set, then
$\mathcal{P}(S) =$ set of all subsets of S
is a set (axiom)

Union, Difference, Intersection

If A, B are sets, then
$A \cup B = \{x | x \in A \text{ or } x \in B\}$
$A \cap B = \{x | x \in A \text{ and } x \in B\}$
$A | B = \{x | x \in A \text{ and } x \notin B\}$
are all set.

Problem of the course.
Given a finite set, how many elements are in it?

Examples of sets

$\mathbb{N} = \{0,1,2,3,4, \ldots\}$
Natural numbers
$\mathbb{Z}$ integers
$\mathbb{Q}$ rationals
$\mathbb{R}$ reals
$\mathbb{C}$ complex numbers

$N_n = \{1,2,3, \ldots, n\} = [n], \quad n \in \mathbb{N}$
$N_0 = \emptyset = \{\ \}$
$N_n = \{\ k \ | k \in \mathbb{N}, \quad 1 \leq k \leq n\} = \{k \in \mathbb{N} | \ 1 \leq k \leq n\}$

Set of perfect matchings in G where G is a graph.

Set of full binary trees.

S, T, sets. $\mathcal{F}(S, T) =$ set of functions from S to T

These are not sets

- The "set" of all sets
  This is not a set. Suppose it were set. Call it S.
  Form the subset $y = \{\ A \mid A \in S \text{ and } A \notin A\}$
  Is $y \in y$?
  Suppose $y \in y \Rightarrow y \in S$ and $y \notin y$
  Suppose $y \notin y$. $y \in S$ and $y \notin y \Rightarrow y \in y$
- "Set" of all finite sets
- "Set" of all trees
- "Set" of all graphs
  - Why? Because trees/graphs have underlying vertex sets.

Bijections & Set Operations

September-11-13 10:33 AM

Bijection

S, T sets
$f: S \to T$ function

- f is **injective** if for every $s, s' \in S$, if $f(s) = f(s')$ then $s = s'$
- f is **surjective** if for every $t \in T$, there exists $s \in S$ such that $f(s) = t$
- f is **bijective** if it is surjective and injective

Note

f is a bijection $\Leftrightarrow$ f has an inverse $f^{-1}: T \to S$,
Inverse: $f\left(f^{-1}(t)\right) = t \ \forall t \in T$ and $f^{-1}\left(f(s)\right) = s \ \forall s \in S$

When you see "bijection" think "two ways of thinking about the same thing".

Equicardinal

If there is a bijection $f: S \to T$, we say S and T are equicardinal and write $S \rightleftarrows T$.
$\rightleftarrows$ is an equivalence relation.

Finite

We say S is a finite set if $S \rightleftarrows N_n$ for some $n \in \mathbb{N}$
We say n is the cardinality of S and write
$n = |S| = \#S$

Theorem 1

A finite set has only one cardinality.

Theorem 2

Two sets are equicardinal $\Leftrightarrow$ they have the same cardinality.

Statement

If S is finite, we can express $\#S = \sum_{s \in S} 1$

If $f: S \to T$ is any function, $t \in T$
Write $f^{-1}(\{t\}) = \{s \in S \mid f(s) = t\}$
Common practice: drop annoying brackets and write this as $f^{-1}(t)$ for $f^{-1}(\{t\})$
(Required: Feel guilty)

Proposition 1

If S, T finite sets
$f: S \to T$ any function

$$\#S = \sum_{t \in T} \#f^{-1}(t)$$

Proposition 2

If S, T are finite sets
$\#(S, T) = (\#S) \cdot (\#T)$
More generally,
$\#(S_1 \times S_2 \times \cdots \times S_k) = (\#S_1) \times (\#S_2) \times \cdots \times (\#S_k)$

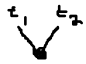
Proposition

If X, Y are finite sets
$\#\mathcal{F}(X, Y) = (\#Y)^{\#X}$

Proof

See notes (like example)

Example

Let B be the set of full binary trees
$f: B \times B \to B$
$(t_1, t_2) \mapsto$ 

Is this a bijection? No - but very close
Injection? Yes
Can figure out t_1 and t_2 uniquely from $f(t_1, t_2)$
$t_1 =$ left branch
$t_2 =$ right branch
Surjection?
The tree $\cdot$ is not in the image

Can be "fixed"
$f: B \times B \cup \{\cdot\} \to B$

Proof of Theorem 1

In HW A1 proved $N_n \rightleftarrows N_m \Leftrightarrow n = m$
If $G \rightleftarrows N_n$ and $G \rightleftarrows N_m$ then since $\rightleftarrows$ is an equivalence relation, $N_n \rightleftarrows N_m$ so $n = m$. $\therefore G$ has only one cardinality.

Proof of Theorem 2

Suppose $G \rightleftarrows N_n, H \rightleftarrows N_m,$ and $G \rightleftarrows H$
Then $N_n \rightleftarrows N_m \Rightarrow n = m$ so G and H have the same cardinality.

Proof of Proposition 1

$$\#S = \sum_{s \in S} 1 = \sum_{t \in T} \sum_{s \in f^{-1}(t)} 1 = \sum_{t \in T} \left(\sum_{s \in f^{-1}(t)} 1 \right) = \sum_{t \in T} \#f^{-1}(t)$$

■

Proof of Proposition 2

Let $f: S \times T \to T$
$f(s, t) = t$

For each $t \in T, f^{-1}(t) = \{(s, t) \mid s \in S\} \rightleftarrows S$

$$\#(S \times T) = \sum_{t \in T} \#f^{-1}(t) = \sum_{t \in T} \#S = \#S \cdot \sum_{t \in T} 1 = (\#S) \cdot (\#T)$$

Example

What is $\#\left(\mathcal{F}(N_m, N_n)\right) = n^m$
Define a function
$\phi: \mathcal{F}(N_m, N_n) \to N_n \times N_n \times \cdots \times N_n$
$f \mapsto \left(f(1), f(2), \ldots, f(m)\right)$
ϕ is a bijection
$\therefore \#\mathcal{F}(N_m, N_n) = \#(N_n)^m = n^m$

This means we can associate functions with sets of tuples (the values of the function)

Set Operations Continued

September-13-13 10:32 AM

Power Set

$\mathcal{P}(V)$ = set of subsets of V

Proposition 1

If V is a finite set,
$\#\mathcal{P}(V) = 2^{\#V}$

Notation

Let $A_1, \dots, A_m$ be finite sets
For each $S \subseteq N_m, S \neq \emptyset$ define

$$A_S = \bigcap_{i \in S} A_i$$

Theorem (Principle of Inclusion-Exclusion)

$$\#(A_1 \cup A_2 \cup \dots \cup A_m) = \sum_{\emptyset \neq S \subseteq N_m} (-1)^{\#S-1} \#A_S$$

Proof in Notes

Proof of Proposition 1

If $S \in \mathcal{P}(V)$ $(S \subseteq V)$
then we can represent it by its **characteristic function**:
$\chi_S: V \to \{0,1\}$

$$\chi_S = \begin{cases} 0 & \text{if } v \notin S \\ 1 & \text{if } v \in S \end{cases}$$

Note: $S = \chi_S^{-1}(1)$

Thus we have
$\mathcal{P}(V) \rightleftarrows \mathcal{F}(V, \{0,1\})$
☐ Check details
$S \mapsto \chi_S$

$\#\mathcal{P}(V) = \#\mathcal{F}(V, \{0,1\}) = 2^{\#V}$

Putting this together with the bijection between functions and tuples,
Subsets of $N_m \rightleftarrows$ binary string of length m

Example

$m = 5$
$\{1,3\} \mapsto 10100$

Proposition

$\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$
$\#(X \cup Y \cup Z) = \#X + \#Y + \#Z - \#(X \cap Y) - \#(X \cap Z) - \#(Y \cap Z) + \#(X \cap Y \cap Z)$

Proof

Draw pictures

Example

$A_{\{1,3,4\}} = A_1 \cap A_3 \cap A_4$

Example of Principle of Inclusion-Exclusion

$m = 3$, $A_1 = X, A_2 = Y, A_3 = Z$

S	$(-1)^{\#S-1}$	A_S	$(-1)^{\#S-1}\#A_S$
$\{1\}$	1	X	$+\#X$
$\{2\}$	1	Y	$+\#Y$
$\{3\}$	1	Z	$+\#Z$
$\{1,2\}$	-1	$X \cap Y$	$-\#(X \cap Y)$
$\{1,3\}$	-1	$X \cap Z$	$-\#(X \cap Z)$
$\{2,3\}$	-1	$Y \cap Z$	$-\#(Y \cap Z)$
$\{1,2,3\}$	1	$X \cap Y \cap Z$	$+\#(X \cap Y \cap Z)$

Note the similarity to $1 - (1 - y_1)(1 - y_2)(1 - y_3)$
$= y_1 + y_2 + y_3 - y_1 y_2 - y_1 y_3 - y_2 y_3 + y_1 y_2 y_3$

More generally, let $y_1 \dots, y_m$ be variables.
If $S \subseteq N_m, S \neq \emptyset$

let $y^S = \prod_{i \in S} y_i$ (if you like, $y^\emptyset = 1$)

Then

$$1 - (1 - y_1)(1 - y_2) \dots (1 - y_m) = 1 - \prod_{i=1}^{m} (1 - y_i) = \sum_{\emptyset \neq S \subseteq N_m} (-1)^{\#S-1} y^S$$

Interpretation

Suppose $A_1, \dots, A_m \subseteq X$
Pick an element $x \in X$ at random.
Let $y_i = \text{Prob}[X \in A_i]$ for $i = 1, \dots, m$
$= \frac{|A_i|}{|X|}$
Assume that the events $x \in A_1, x \in A_2, \dots, x \in A_m$ are mutually independent.

Mutual independences means that
$$\frac{|A_S|}{|X|} = \prod_{i \in S} \frac{|A_i|}{|X_S|} = y^S$$

In I-E formula
$$\frac{\text{LHS}}{|X|} = \frac{|A_1 \cup \dots \cup A_m|}{|X|} = \text{Prob}[x \in A_1 \text{ or } x \in A_2 \text{ or } \dots \text{ or } x \in A_m]$$
$$= 1 - \text{Prob}[x \notin A_1 \text{ and } x \notin A_2 \text{ and } \dots \text{ and } x \notin A_m] = 1 - (1 - y_1)(1 - y_2) \dots (1 - y_m)$$
$$\frac{\text{RHS}}{|X|} = \sum_{\emptyset \neq S \subseteq N_m} (-1)^{|S|-1} \frac{|A_S|}{|X|} = \sum_{\emptyset \neq S \subseteq N_m} (-1)^{|S|-1} y^S$$

How to use it: Think of $A_1, \dots, A_m$ as all the bad things that could happen.

Example

If $n \in \mathbb{N}$, the Euler totient of n is

$\phi(n) = \#\{ k \in N_n \mid \gcd(k, n) = 1 \}$
e.g. $\phi(6) = \#\{1,5\} = 2$

Write $n = p_1^{c_1} p_2^{c_2} \cdots p_m^{c_m} \leftarrow$ prime factorization
Bad things: prime factor in common
Let $A_i = \{k \in N_n \mid p_i \text{ divides } k\}$
Then $A_1 \cup A_2 \cup \cdots \cup A_m = \{k \in N_n \mid \gcd(k, n) \neq 1\}$
$\phi(n) = n - \#(A_1 \cup A_2 \cup \cdots \cup A_m)$

Now use Inclusion-Exclusion Formula
For $S \subseteq N_m$, $\quad b \in A_S \Leftrightarrow \left(\prod_{i \in S} p_i \right)$ divides b
(e.g. $b \in A_1 \cap A_2 \Leftrightarrow p_1 p_2$ divides b)
$|A_S| = \frac{n}{\prod p_i} = n \cdot \left(\prod_{i \in S} \frac{1}{p_i} \right)$
Let $y_i = \frac{1}{p_i}$ then $|A_S| = n \cdot y^S$
So By I-E
$|A_1 \cup \cdots \cup A_m| = n \cdot \left(\sum_{\emptyset \neq S \subseteq N_n} (-1)^{|S|-1} y^S \right) = n\big(1 - (1 - y_1)(1 - y_2) \dots (1 - y_m)\big)$

Permutations

September-16-13 10:33 AM

Permutation
A permutation of length n is a bijection $\sigma: N_n \to N_n$.
The set of all permutations of length n is denoted S_n
Occasionally denoted $\mathfrak{S}_n$

Word Notation (a.k.a. one line notation)
Express σ as $a_1 a_2 a_3 \dots a_n$ where $a_i = \sigma(i)$
Don't use brackets in word notation. If necessary, use square brackets.

Example: 25143 is word notation for the permutation $\sigma: N_5 \to N_5$
$\sigma(1) = 2, \quad \sigma(2) = 5, \quad \sigma(3) = 1, \quad \sigma(4) = 4, \quad \sigma(5) = 5$

Theorem
$|S_n| = n!$

Permutation
Let X be a finite set. A permutation of X is a bijection $\sigma: X \to X$
Let S_X be the set of all permutations of X.

Corollary
If $|X| = n$, then $|S_X| = n!$

Informal Proof of Theorem $|S_n| = n!$
To specify an element $a_1 a_2 \dots a_n \in S_n$, there are n ways to choose a_1. For each of these there are $n-1$ ways to choose a_2 and so on.

Advantages of this informal proof
- Easy to understand
- Short and sweet

Disadvantages
- Hard to tell if there are any holes in the reasoning.
- Does not appeal to proven facts (e.g. chapter 1)
- Doesn't give an algorithm for listing all permutations.
- By defining a more formal proof get extra info.

Strong Induction CONSPIRACY
Goal: Proof $P(k)$, for $k = 1,2,3,\dots$
~~Base case: Prove $P(1)$~~
Inductive Hypothesis: Assume $P(1), \dots, P(k-1)$
Inductive Step: Deduce $P(k)$ from the inductive hypothesis.

There is no need for the base case if your inductive step can deduce $P(1)$ from the inductive hypothesis of nothing.

Proof of Theorem $|S_n| = n!$
Let $Q_n = N_n \times N_{n-1} \times N_{n-2} \times \dots \times N_2 \times N_1$
then $|Q_n| = n!$
We'll prove that $S_n \rightleftarrows Q_n$
by giving maps $I_n: S_n \to Q_n$
$J_n: Q_n \to S_n$
and prove that they are mutually inverse.

Function: $I_n: S_n \to Q_n$
Input: $a_1 a_2 \cdots a_n \in S_n$
```
repeat with i from 1 to n:
    let r_i = |{j ∈ ℕ | i < j ≤ n and a_i > a_j}|
end
```
Output: $(r_1 + 1, r_2 + 1, \dots, r_n + 1)$

Example:
$\sigma = 3\,1\,5\,7\,6\,4\,2$
$I_7(\sigma) = (3, 1, 3, 4, 3, 2, 1)$
We need to check that $I_n(\sigma) \in Q_n$ for all $\sigma \in S_n$
An element of Q_n is a integer tuple $(h_1, \dots, h_n)$ such that $1 \le h_i \le n - i + 1$

So we need to show that $1 \le 1 + r_i \le n - i + 1 \Leftrightarrow 0 \le r_i \le n - i$
- $r_i \in \mathbb{N}$ because r_i is the cardinality of a set. Hence $r_i \ge 0$
- $r_i = |\{j \in \mathbb{N} \mid i < j \le n \text{ and } a_i > a_j\}|$
  $\{j \in \mathbb{N} \mid i < j \le n \text{ and } a_i > a_j\} \subseteq \{j \in \mathbb{N} \mid i < j \le n\}$
  Since $|\{j \in \mathbb{N} \mid i < j \le n\}| = n - i$, $r_i \le n - i$

Function $J_n: Q_n \to S_n$
Input: $(h_1, \dots, h_n) \in Q_n$
```
repeat for i from 1 to h
    let b_i be the (h_i)^th smallest element of N_n\{b_1,...,b_{i-1}}
end repeat
```
Output: $b_1 b_2 \dots b_n$

Note
- $b_i \ne b_j$ for any $j < i$
- Hence $|(N_n \setminus \{b_1, \dots, b_{i-1}\})| = n - i + 1$
- Since $1 \le h_i \le n - i + 1$, there is an $(h_i)^{\text{th}}$ smallest element.
- Also, this shows $b_1 b_2 \dots b_n \in S_n$ since we've listed the elements of N_n in some order.

Example
$J_7(5, 1, 3, 4, 2, 1, 1) = 5\,1\,4\,7\,3\,2\,6$

Finally Prove
1) $J_n(I_n(\sigma)) = \sigma$ for all $\sigma \in S_n$
2) $I_n(J_n(\rho)) = \rho$ for all $\rho \in Q_n$

Proof of 1
Let $\sigma = a_1 \dots a_n \in S_n$
$I_n(\sigma) = (r_1 + 1, \dots, r_n + 1)$
Let $J_n(r_1 + 1, \dots, r_n + 1) = b_1 b_2 \dots b_n$
We must show that $a_i = b_i$ for all i
By definition, $r_i = |\{j \in \mathbb{N} \mid i < j \le n \text{ and } a_i > a_j\}|$
$\therefore a_i$ is the $(r_i + 1)^{\text{th}}$ smallest element of $\{a_i, a_{i+1}, \dots, a_n\}$
Now proceed by strong induction

Induction hypothesis.
Assume $a_1 = b_1, a_2 = b_2, \dots, a_{i-1} = b_{i-1}$

Induction step:
$N_n \setminus \{b_1, \dots, b_{i-1}\} = N_n \setminus \{a_1, \dots, a_{i-1}\} = \{a_i, a_{i+1}, \dots, a_n\}$
(Since $\{a_1, \dots, a_n\} = N_n$ by definition of a permutation)
$\therefore$ by definition of J_n
$b_i = (r_i + 1)^{\text{th}}$ smallest element of $\{a_i, a_{i+1}, \dots, a_n\}$
$\therefore a_i = b_i$ as required.

Proof of 2
Homework
This completes the proof. ■

We're using the fact that a function $f: X \to Y$ is a bijection
$\Leftrightarrow$ it has an inverse $g: Y \to X$ s.t. $g(f(x)) = x \; \forall x \in X$ and $f(g(y)) = y \; \forall y \in Y$

- [ ] Recommended exercise: prove this.

Proof of Corollary

We'll show that $S_X \rightleftarrows S_n$
Since $|X| = n$, there exists a bijection f: $X \to N_n$. For $\sigma \in S_X$ defined
$\alpha(\sigma) = f \circ \sigma \circ f^{-1}: N_n \to N_n$
$\alpha(\sigma): \xrightarrow{f^{-1}} X \xrightarrow{\sigma} X \xrightarrow{f} N_n$
Since $\alpha(\sigma)$ is a composition of bijections, it is a bijection. $\therefore \alpha(\sigma) \in S_n$
In other words: $\alpha: S_X \to S_n$
Similarly define $\beta: S_n \to S_X$ by $\beta(\tau) = f^{-1} \circ \tau \circ f$
You can check that α and β are mutually inverse. ■

See also the bit about $\mathcal{F}(X, Y)$ in Ch. 1.

Subsets

September-18-13 10:50 AM

Theorem

$|\mathcal{B}(n,k)| = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Definition

If X is finite set, define
$\mathcal{B}(X,k)$ = set of all k-element subsets of X

Corollary

If $|X| = n$ then $|\mathcal{B}(X,k)| = \binom{n}{k}$

Proof

See notes for details of $\mathcal{B}(X,k) \rightleftarrows \mathcal{B}(n,k)$

Binomial Theorem

For $n \in \mathbb{N}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} n^k$$

Derangement

A permutation of $\sigma \in S_n$ is called a derangement if $\sigma(i) \neq i$for all $i = 1, \dots, n$
Let $\mathcal{D}_n \subseteq S_n$ denote the set of all derangements of length n.

Let $\mathcal{B}(n,k)$ = set of all k-element subsets of N_n
e.g.
$\mathcal{B}(4,2) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$
abuse notation $= \{12,13,14,23,24,34\}$

Proof of Theorem

We will show that
$\mathcal{B}(n,k) \times S_k \times S_{n-k} \rightleftarrows S_n$
this shows that
$|\mathcal{B}(n,k)| \cdot |S_k| \cdot |S_{n-k}| = |S_n|$
which gives use the result since $|S_m| = m!$

Strategy to show $|X| = |Y|$

1. define $f: X \to Y$
2. define $g: Y \to X$
3. show that $f(g(y)) = y \;\; \forall y \in Y$
4. show that $g(f(x)) = x \;\; \forall x \in X$

Step 1

To do this, construct bijections
Useful subroutine:
Let R_m be the set of sequence of pairwise distinct positive integers $a_1, \dots, a_m$ (of length m)
E.g. $1\,5\,2 \in R_3,\;\; 1\,2\,1 \notin R_3$
Function: $P_m: R_m \to S_m$
Input $a_1, \dots, a_m$
```
repeat with i from 1 to m
    let b_i = |{j ∈ N_m | a_i ≥ a_j}|
end repeat
```
Output: $b_1 b_2 \dots b_m$

$P_m(5\,2\,9\,4\,6) = 3\,1\,5\,2\,4$
Note: if $\beta = P_i(a_1, \dots, a_m)$ then a_i is the $b(i)^{\text{th}}$ smallest element of $\{a_1, \dots, a_m\}$

Function: $\Psi_{n,k}: S_n \to \mathcal{B}(n,k) \times S_k \times S_{n-k}$
Input: $a_1 a_2 \dots a_n$
```
Let A = {a_1, a_2, ..., a_k}
    β = P_k(a_1, a_2, ..., a_k)
    γ = P_{n-k}(a_{k+1}, ..., a_n)
```
Output: (A, β, γ)

Example

$\Psi_{7,4}(3\,6\,7\,1\,4\,5\,2) = (\{1,3,6,7\}, 2\,3\,4\,1, 2\,3\,1)$

Step 2

Function $\Phi_{n,k}: \mathcal{B}(n,k) \times S_k \times S_{n-k} \to S_n$
Input: (A, β, γ)
Sort A as $s_1 < s_2 < \cdots < s_k$
sort $N_n \setminus A$ as $t_1 < t_2 < \cdots < t_{n-k}$
```
repeat with i from 1 to k
    let c_i = s_{β(i)}
end repeat
repeat with j from 1 to n-k
    let c_{k+j} = t_{γ(j)}
end repeat
```

Example

$\Phi_{7,4} = (\{1,3,6,7\}, 2\,3\,4\,1, 2\,3\,1)$
$s_1 = 1, s_2 = 3, s_3 = 6, s_4 = 7$
$t_1 = 2, t_2 = 4, t_3 = 5$

$c_1 = s_2 = 3$
$c_2 = s_3 = 6$
$c_3 = s_4 = 7$
$c_4 = s_1 = 1$

$c_5 = t_2 = 4$
$c_6 = t_3 = 5$
$c_7 = t_1 = 2$

Final steps

show:

1) $\Phi_{n,k}\left(\Psi_{n,k}(\sigma)\right) = \sigma \; \forall \sigma \in S_n$
2) $\Psi_{n,k}\left(\Phi_{n_k}(A,\beta,\gamma)\right) = (A,\beta,\gamma) \; \forall (A,\beta,\gamma) \in \mathcal{B}(n,k) \times S_k \times S_{n-k}$

Proof of 1

Let $\sigma = a_1 a_2 \dots a_n$
Let $\Psi_{n,k}(\sigma) = (A, \beta, \gamma)$
Let $\Phi_{n,k}(A,\beta,\gamma) = c_1 c_2 \dots c_n$
We must show that $a_i = c_i$ for $i = 1, \dots, n$

In the algorithm for $\Phi_{n,k}$, c_i is the $\beta(i)^{\text{th}}$ smallest element of A. But as we observed, in the

definition of P_n
a_i is the $\beta(i)^{\text{th}}$ smallest element of A since $\beta = P_n(a_1, \ldots, a_n)$
$\therefore a_i = c_i$ for $i = 1, \ldots, k$
Similarly, c_{k+j} is the $\gamma(j)^{\text{th}}$ smallest element of $N_n \backslash$A and so is a_{k+j}
$\therefore a_{k+j} = c_{k+j}$ for $k = 1, \ldots, n-k$

Proof of 2

Exercise

Proof of Binomial Theorem

From the identity

$$(1+y_1)(1+y_2)\cdots(1+y_n) = \sum_{S\subseteq N_n} y^S$$

Set $y_1 = y_2 = \cdots = y_n = X$
LHS $\rightarrow (1+X)^n$

RHS $\rightarrow \sum_{S\subseteq N_n} X^{|S|}$

Since $y^s = \prod_{i\in S} y_i \rightarrow \prod_{i\in S} X = X^{|S|}$

$$\sum_{S\subseteq N_n} X^{|S|} = \sum_{k=0}^{n}\sum_{S\in\mathcal{B}(n,k)} X^{|S|} = \sum_{k=0}^{n}\sum_{S\in\mathcal{B}(n,k)} X^k = \sum_{k=0}^{n} X^k \sum_{S\in\mathcal{B}(n,k)} 1 = \sum_{k=0}^{n} X^k |\mathcal{B}(n,k)| = \sum_{k=0}^{n}\binom{n}{k} X^k$$

Example: Derangements

What is $|\mathcal{D}_n|$?
Bad things that could happen:
$\sigma(1) = 1, \sigma(2) = 2, \ldots$
Let $A_i = \{\sigma \in S_n \mid \sigma(i) = i\}$
Then $\mathcal{D}_n = S_n \setminus \{A_1 \cup A_2 \cup \cdots \cup A_n\}$
For $\emptyset \neq S \subseteq N_n$

$$A_S = \bigcap_{i\in S} A_i = \{\sigma \in S_n \mid \sigma(i) = i\ \forall i \in S\}$$

e.g. $A_{\{1,2,3\}} = A_1 \cap A_2 \cap A_3 = \{\sigma \in S_n \mid \sigma(1) = 1, \sigma(2) = 2, \text{and } \sigma(3) = 3\}$

If $|S| = k$, this means we've fixed k values of any permutation $\sigma \in A_S$
The remaining values are specified (and specify) a permutation of $N_n \setminus S$
i.e. $A_S \rightleftarrows S_{N_n \setminus S} \Rightarrow |A_S| = (n-k)!$
Now compute

$$|\mathcal{D}_n| = |S_n \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)| = n! - |A_1 \cup \cdots \cup A_n| = n! - \sum_{\emptyset\neq S\subseteq N_n} (-1)^{|S|-1}|A_S|$$

$$= n! - \sum_{k=1}^{n}\sum_{S\in\mathcal{B}(n,k)} (-1)^{|S|-1}|A_S| = n! - \sum_{k=1}^{n}\sum_{S\in\mathcal{B}(n,k)} (-1)^{k-1}(n-k)!$$

$$= n! - \sum_{k=1}^{n} (-1)^{k-1}(n-k)! \sum_{S\in\mathcal{B}(n,k)} 1 = n! - \sum_{k=1}^{n} (-1)^{k-1}(n-k)!\binom{n}{k} = \cdots = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

Note:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = \frac{1}{e}$$

So if n is large, $\frac{|\mathcal{D}_n|}{n!} \approx \frac{1}{e}$

Proof of Binomial Theorem

From the identity

$$(1+y_1)(1+y_2)\cdots(1+y_n) = \sum_{S\subseteq N_n} y^S$$

Set $y_1 = y_2 = \cdots = y_n = X$
LHS $\rightarrow (1+X)^n$

RHS $\rightarrow \sum_{S\subseteq N_n} X^{|S|}$

Since $y^s = \prod_{i\in S} y_i \rightarrow \prod_{i\in S} X = X^{|S|}$

$$\sum_{S\subseteq N_n} X^{|S|} = \sum_{k=0}^{n}\sum_{S\in\mathcal{B}(n,k)} X^{|S|} = \sum_{k=0}^{n}\sum_{S\in\mathcal{B}(n,k)} X^k = \sum_{k=0}^{n} X^k \sum_{S\in\mathcal{B}(n,k)} 1 = \sum_{k=0}^{n} X^k |\mathcal{B}(n,k)| = \sum_{k=0}^{n}\binom{n}{k} X^k$$

Lattice Paths

September-23-13 10:34 AM

Lattice Paths

For $a, b \in \mathbb{N}$, let $\mathcal{L}(a,b)$ be the set of all lattice paths from $(0,0)$ to (a,b) in the plane $\mathbb{Z} \times \mathbb{Z}$ which consist of steps either

east: $(x,y) \to (x+1,y)$ or

north: $(x,y) \to (x,y+1)$

Notation: Coproduct

$\coprod_{j=0}^{b} A_i$ means $\bigcup_{j=0}^{b} A_i$ where $A_1, \dots, A_b$ are pairwise disjoint.

Set Partition

Let X be a finite set.

A set partition of X is a finite set $\Pi = \{B_1, \dots, B_k\}$ such that

- $\emptyset \neq B_i \subseteq X$ for $i = 1, \dots, k$
- $B_i \cap B_j = \emptyset$ for $i \neq j$
- $B_1 \cup \cdots \cup B_k = X$

Stirling Numbers of The Second Kind

a.k.a Stirling Subset Numbers

Let $\Pi(n,k)$ be the set of all set partitions of N_n into k subsets.

$S(n,k) := |\Pi(n,k)|$

$S(n,k)$ is a Stirling number of the second kind.

- $\Pi(n,k)$ is a set of sets of sets.
- Elements of $\pi \in \Pi(n,k)$ are set partitions.
- Elements $B \in \pi$ are subsets of N_n

Recursion Property of Stirling Numbers

$S(0,0) = 1$ (The empty set is a set partition of $\emptyset$ so $\Pi(0,0) = \{\emptyset\}$)

$S(0,k) = 0$ for $k \geq 1$

$S(n,0) = 0$ for $n \geq 1$

$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

Reminiscent of $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

What is $|\mathcal{L}(a,b)|$?

$|\mathcal{L}(a,b)| = \binom{a+b}{a} = \binom{a+b}{b}$

A path can be represented as a sequence $P = s_1 s_2 \dots s_{a+b}$ where $s_i \in \{E, N\}$.

$P \in \mathcal{L}(a,b) \iff a$ of its steps are E and b of its steps are N.

Gives a bijection $\mathcal{L}(a,b) \rightleftarrows \mathcal{B}(a+b,b)$ since

$s_1 s_2 \dots s_{a+b} \mapsto \{i \in N_{a+b} \mid s_i = N\}$

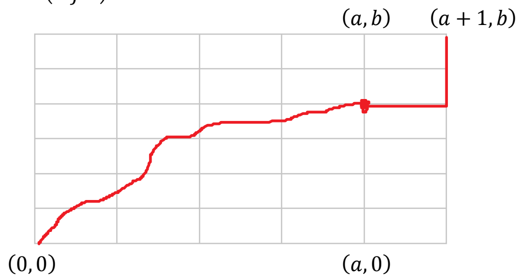
Example

Prove that for $a, b \in \mathbb{N}$

$$\binom{a+1+b}{b} = \sum_{j=0}^{b} \binom{a+j}{j}$$

LHS $= |\mathcal{L}(a+b,b)|$

$\binom{a+j}{j} = |\mathcal{L}(a,j)|$



Any path from $(0,0)$ to $(a+1,b)$ is a path from $(0,0)$ to (a,j) followed by $EN^{b-j} = ENN \cdots N$ for some unique $j \in \{0,1,\dots,b\}$

We have a bijection

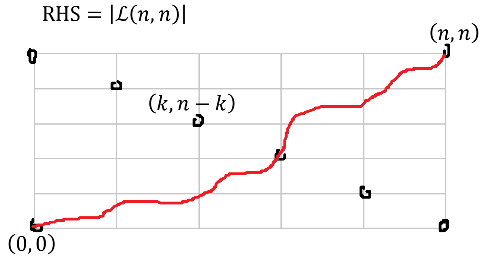
$$\mathcal{L}(a+1,b) \rightleftarrows \coprod_{j=0}^{b} \mathcal{L}(a,j) \times \{EN^{b-j}\}$$

$$\text{LHS} = |\mathcal{L}(a+1,b)| = \left| \coprod_{j=0}^{b} \mathcal{L}(a,j) \times \{EN^{b-j}\} \right| = \text{RHS}$$

Example

Prove for $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

RHS $= |\mathcal{L}(n,n)|$



Each path passes through exactly one of the points $(k, n-k)$ for some k.

$$\coprod_{i=0}^{n} \mathcal{L}(k,n-k) \times \mathcal{L}(n-k,k) \rightleftarrows \mathcal{L}(n,n)$$

Example Set Partition

$\{\{1,3,6\},\{2,4\},\{5\},\{7\}\}$

is a set partition of N_7 into 4 subsets ($k = 4$)

Proof of Recursion Properties of Stirling Numbers

$\Pi'(n,k) \subseteq \Pi(n,k)$

$\Pi'(n,k) = \{\pi \in \Pi(n,k) \mid \{n\} \in \pi\}$

i.e. $\{n\}$ appears in Π

n appears alone in a subset in Π.

Let $\Pi''(n,k) = \Pi(n,k) \setminus \Pi'(n,k)$

Claim: $\Pi'(n,k) \rightleftarrows \Pi(n-1,k)$

$\pi \mapsto \pi \setminus \{\{n\}\}$

$\mu \cup \{\{n\}\} \leftarrow \mu$

This shows that $|\Pi'(n,k)| = S(n-1,k)$

Example

$$\Pi(4,2) = \left\{ \begin{matrix} \{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\},\{\{1\},\{2,3,4\}\}, \\ \{\{2\},\{1,3,4\}\},\{\{3\},\{1,2,4\}\},\{\{4\},\{1,2,3\}\} \end{matrix} \right\}$$

$\Pi'(4,2) = \{\{\{4\},\{1,2,3\}\}\}$

$\Pi''(4,2) = \{ \text{rest} \}$

Last time we saw $\Pi'(n,k) \rightleftarrows \Pi(n-1,k-1)$

Also $\Pi''(n,k) \to \Pi(n-1,k)$

$\pi \mapsto \{ B \setminus \{n\} \mid B \in n\}$

e.g. $(n=7)$ $\pi = \{\{1,3,7\},\{2,4,5,6\}\} \mapsto \{\{1,3\},\{2,4,5,6\}\}$

The preimage of any $u \in \Pi(n-1,k)$ under this map has exactly k elements.

(there are k ways we can put n back into one of the subsets)

$|\Pi''(n,k)| = k \cdot |\Pi(n-1,k)| = k \cdot S(n-1,k)$

Since we had $\Pi(n,k) = \Pi'(n,k) \sqcup \Pi''(n,k)$

$S(n,k) = S(n-1,k-1) + kS(n-1,k)$

Polynomial Identities

September-25-13 10:50 AM

Theorem

Let $p(y)$ and $q(y)$ be polynomials. If there are infinitely many natural numbers $n \in \mathbb{N}$ such that $p(n) = q(n)$, then $p(y) \equiv q(y)$ as polynomials.

Proof

For any polynomial $f(y)$, if $f(y) \not\equiv 0$, then $f(y)$ has only finitely many roots S.
Let $f(y) = p(y) - q(y)$
If $p(n) = q(n)$ for infinitely many $n \Rightarrow f(n) = 0$
$\therefore f(y)$ must be the zero polynomial. ■

Example

With the formula

$$\binom{y}{k} = \frac{y(y-1)(y-2)\cdots(y-k+1)}{k!}$$

$\binom{y}{k}$ is a polynomial of degree k in y ($k \in \mathbb{N}$ is constant)

Consider

$$\binom{y+1+b}{b} = \sum_{j=0}^{b} \binom{y+j}{j}, \qquad \text{for } b \in \mathbb{N}$$

Last time we proved this is true if $y = a \in \mathbb{N}$
LHS: polynomial in y
RHS: sum of $b+1$ polys $\Rightarrow$ polynomial
This is true as a polynomial identity.

- [ ] See exercise 3.9 for something that looks similar but doesn't work this way

Multivariable Version (Example)

If $p(y_1, \ldots, y_m)$ and $q(y_1, \ldots, y_m)$ polynomials in m variables and
$p(n_1, n_2, \ldots, n_m) = p(n_1, \ldots, n_m)$
for $(n_1, \ldots, n_m) \in S_1 \times \cdots \times S_n$ where $S_i \subseteq \mathbb{N}$ infinite for all i. Then $p \equiv q$

Generating Functions

September-25-13 11:08 AM

Notation

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r$
If $x = (x_1, \ldots, x_r)$ is a sequence of (pairwise commuting) indeterminates. We write
$\boldsymbol{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$
If $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r,\ \beta = (\beta_1, .., \beta_r) \in \mathbb{N}^r$
$\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_r + \beta_r)$

If $\alpha_i \le \beta_i$ for all i, we'll write $\alpha \le \beta$. Hence $\boldsymbol{x}^\alpha$ divides $\boldsymbol{x}^\beta \Leftrightarrow \alpha \le \beta$

Weight Function

If S is a set, a function $\omega: S \to \mathbb{N}^r$ is a **weight function** if $|\omega^{-1}(\alpha)|$ is finite for all $\alpha \in \mathbb{N}^r$

In this setup, define the generation function of S with respect to ω to be

$$\Phi_S^\omega(x) = \sum_{s \in S} \boldsymbol{x}^{\omega(s)}$$

Hence $x = (x_1, \ldots, x_r)$ is a sequence of r pairwise commuting indeterminates.

Example

You have two ordinary 6-sided dice. (Assume distinguishable)
How many ways to roll a 9?

Mathematically

A die roll is modelled as an element of N_6
Determine $|\{(a,b) \in N_6 \times N_6 \mid a + b = 9\}|$
More generally, $|\{(a,b) \in N_6 \times N_6 \mid a + b = n\}|$

To solve, consider

$$\sum_{(a,b) \in N_6 \times N_6} x^{a+b}$$

We get x^n for every $a + b = n$. Our answer is therefore the coefficient of x^n in this.

$$= \sum_{a \in N_6} \sum_{b \in N_6} x^{a+b} = \sum_{a \in N_6} \sum_{b \in N_6} x^a x^b = \left(\sum_{a \in N_6} x^a\right)\left(\sum_{b \in N_6} x^b\right) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

$$= \left(\frac{x - x^7}{1 - x}\right)^2$$

Answer: $[x^n]\left(\frac{x - x^7}{1 - x}\right)^2$

$N_6 \times N_6$ is out set of objects
$n = a + b$ is the weight of $(a,b) \in N_6 \times N_6$

Advantages

Encodes our answer in a way
☐ ... (Get the rest of this from someone)

Example

A composition of n with k parts is a k-tuple $(c_1, \ldots, c_k)$ where $c_1, c_2, \ldots, c_k$ are positive integers and $c_1 + c_2 + \cdots + c_k = n$

How many compositions of n with k parts?

$$\sum_{(c_1, \ldots, c_k) \in (\mathbb{N}_{\ge 1})^k} x^{c_1 + c_2 + \cdots c_k}, \qquad \text{Answer will be coefficient of } x^n$$

$$= \sum_{c_1 \in \mathbb{N}_{\ge 1}} \sum_{c_2 \in \mathbb{N}_{\ge 1}} \ldots \sum_{c_k \in \mathbb{N}_{\ge 1}} x^{c_1} x^{c_2} \ldots x^{c_k} = \left(\sum_{c_1 \in \mathbb{N} \ge 1} x^{c_1}\right)\left(\sum_{c_2 \in \mathbb{N} \ge 1} x^{c_2}\right) \cdots \left(\sum_{c_k \in \mathbb{N}_{\ge 1}} x^{c_k}\right)$$

$$= (x + x^2 + x^3 + \cdots)^k = \left(\frac{x}{1 - x}\right)^k$$

Answer: $[x^n]\left(\frac{x}{1 - x}\right)^k = \cdots$

$(\mathbb{N}_{\ge 1})^k$ is the set of objects
$n = c_1 + \cdots + c_k$ is the weight of $(c_1, c_2, \ldots, c_k) \in (\mathbb{N} \ge 1)^k$

Weirdness: Two parameters k & n presented very differently

- n disappears from the solution until the end (n is the weight)
- k stays present on every line (k is not the weight)

We'll rectify this particular situation by considering vector valued weight functions.

- No more complicated than integer valued weight functions.
- Can do more.

Example

Let $S = \mathbb{N}$

1) Let $\omega: S \to \mathbb{N},\ \omega(i) = i$
   Then $\Phi_S(x) = \sum_{s \in S} x^{\omega(s)} = \sum_{s \in \mathbb{N}} x^s = x^0 + x^1 + x^2 + \cdots = \frac{1}{1 - x}$
2) Let $\omega: S \to \mathbb{N},\ \omega(i) = 1$
   Trick question. ω is not a weight function because $\omega^{-1}(1)$ infinite
3) Let $\omega: S \to \mathbb{N}^2,\ \omega(i) = (i, 1)$
   What is $\Phi_S(x, y)$?
   $$\Phi_S(x, y) = \sum_{s \in S} (\boldsymbol{x}, \boldsymbol{y})^{w(s)} = \sum_{i \in \mathbb{N}} \boldsymbol{x}^i \boldsymbol{y}^1 = x^o y + x^1 y + x^2 y + \cdots = \frac{y}{1 - x}$$

Example

Let $S = (\mathbb{N}_6)^3$ be the set of outcomes of rolling 3 dice.
(a, b, c) means roll a on die 1, b on die 2, c on die 3.
Define the weight function
$\omega: S \to \mathbb{N}^3, \omega\big((a, b, c)\big) = (a, b, c)$
What is $\Phi_S(x, y, z)$?

$$\Phi_S(x, y, z) = \sum_{(a,b,c) \in (\mathbb{N}_6)^3} (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})^{\omega((a,b,c))} = \sum_{(a,b,c) \in (\mathbb{N}_6)^3} \boldsymbol{x}^a \boldsymbol{y}^b \boldsymbol{z}^c = \sum_{a \in \mathbb{N}_6} \sum_{b \in \mathbb{N}_6} \sum_{c \in \mathbb{N}_6} \boldsymbol{x}^a \boldsymbol{y}^b \boldsymbol{z}^c$$

$$= \left(\sum_{a \in \mathbb{N}_6} \boldsymbol{x}^a\right)\left(\sum_{b \in \mathbb{N}_6} \boldsymbol{y}^b\right)\left(\sum_{c \in \mathbb{N}_6} \boldsymbol{z}^c\right) = D(\boldsymbol{x}) D(\boldsymbol{y})(\boldsymbol{z})$$

where $D(t) = t + t^2 + t^3 + t^4 + t^5 + t^6 = \frac{t - t^7}{1 - t}$

Back to compositions
Let S =set of all compositions

$$S = \bigcup_{k \geq 0} (\mathbb{N}_{\geq 1})^k$$

Ordinary Generating Functions

September-30-13 10:34 AM

Proposition

Let S be a set with a weight function $\omega: S \to \mathbb{N}^\nu$

Then $\Phi_S(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^r} (|\omega^{-1}(\alpha)|)\mathbf{x}^\alpha$

Note: $\omega^{-1}(\alpha)$ is the set of elements in S that have weight α

Coefficient notation:
$[x^\alpha]\Phi_S(x) = |\omega^{-1}(\alpha)|$

However, if we write
$[x^2]\,(x^2+y)(1+y) = [x^2]\,\left((1+y)x^2 + (y+y^2)\right) = 1+y$

$[x^2]\,(x^2+y)(1+y) \neq 1$
Instead, $[x^2y^0](x^2+y)(1+y) = 1$

Weight Preserving Bijection

Let S, T be sets with wight functions

$\omega: S \to \mathbb{N}^r$
$\nu: T \to \mathbb{N}^r$

A function $f: S \to T$ is called a weight perserving bijection if

- f is a bijection
- $\nu(f(s)) = \omega(s)\ \ \forall s \in S$

Proposition 2

Let S and T be as above. There is a weight-preserving bijection $f: S \to T$ if and only if $\Phi_S^\omega(\mathbf{x}) = \Phi_S^\nu(\mathbf{x})$

This is a generalization of
$S \rightleftarrows T$ iff $|S| = |T|$ for finite sets.

Sum Lemma

Let S be a set with weight function $\omega: S \to \mathbb{N}^r$
Suppose $S = S_1 \cup S_2 \cup S_3 \cup \cdots$ (finite or infinite)
where $S_i \cap S_j = \emptyset$ for $i \neq j$

Then $\Phi_S^\omega(\mathbf{x}) = \sum_i \Phi_{S_i}^\omega(\mathbf{x})$

Product Lemma

Let S, T be sets with weight functions $\omega: S \to \mathbb{N}^r, \nu: S \to \mathbb{N}^r$
Define a weight function

$\phi: S \times T \to \mathbb{N}^r$
$\phi(s,t) = \omega(s) + \nu(t)$

$\Phi_{S\times T}^\phi(\mathbf{x}) = \Phi_S(\mathbf{x})\Phi_T(\mathbf{x})$

Proof of Proposition

(Identical to math 239)

$$\text{LHS } \Phi_S(x) = \sum_{s\in S}\mathbf{x}^{\omega(s)} = \sum_{\alpha\in\mathbb{N}^r}\sum_{S\in\omega^{-1}(\alpha)}\mathbf{x}^{\omega(s)} = \sum_{\alpha\in\mathbb{N}^r}\sum_{S\in\omega^{-1}(\alpha)}\mathbf{x}^{\alpha} = \sum_{\alpha\in\mathbb{N}^r}\mathbf{x}^\alpha\left(\sum_{s\in\omega^{-1}(\alpha)} 1\right)$$
$$= \sum_{\alpha\in\mathbb{N}^r}\mathbf{x}^\alpha|\omega^{-1}(\alpha)|$$

■

This tells us that the generating function answers the question: Determine $|\omega^{-1}(\alpha)|$

Proof of Proposition 2

Suppose we have a weight preserving bijection $f: S \to T$
Then

$$\Phi_S^\omega(\mathbf{x}) = \sum_{s\in S}\mathbf{x}^{\omega(s)} = \sum_{t\in f(s)\in T}\mathbf{x}^{\nu(f(s))} = \sum_{t\in T}\mathbf{x}^{\nu(t)} = \Phi_T^\nu(\mathbf{x})$$

Suppose $\Phi_S^\omega(\mathbf{x}) = \Phi_T^\nu(\mathbf{x})$
Then $[\mathbf{x}^\alpha]\Phi_S^\omega(\mathbf{x}) = [\mathbf{x}^\alpha]\Phi_T^\nu(\mathbf{x}), \quad \forall\alpha\in\mathbb{N}^r$
$\Rightarrow |\omega^{-1}(\alpha)| = |\omega^{-1}(\nu)|$
Therefore there is a bijection $f_\alpha: \omega^{-1}(\alpha) \to \nu^{-1}(\alpha)$
Now define a map $f: S \to T$ by saying $f(s) = f_\alpha(s)$ for $\alpha = \omega(s)$
Check that f is injective and surjective.
Also note: If $\omega(s) = \alpha$ then $s \in \omega^{-1}(\alpha) \Rightarrow f_\alpha(s) \in \nu^{-1}(\alpha)$
$\Rightarrow f(s) \in \nu^{-1}(\alpha) \Rightarrow \nu(f(s)) = \alpha$
$\therefore\ \nu(f(s)) = \omega(s)$

Proof of Sum Lemma

☐ Exercise / See notes

Proof of Product Lemma

$$\Phi_{S\times T}(\mathbf{x}) = \sum_{(s,t)\in S\times T}\mathbf{x}^{\phi(s,t)} = \sum_{s\in S}\sum_{t\in T}\mathbf{x}^{\phi(s,t)} = \sum_{s\in S}\sum_{t\in T}\mathbf{x}^{\omega(s)+\nu(t)} = \sum_{s\in S}\sum_{t\in T}\mathbf{x}^{\omega(s)}\mathbf{x}^{\nu(t)}$$
$$= \sum_{s\in S}\mathbf{x}^{\omega(s)}\left(\sum_{t\in T}\mathbf{x}^{\nu(t)}\right) = \left(\sum_{s\in S}\mathbf{x}^{\omega(s)}\right)\left(\sum_{t\in T}\mathbf{x}^{\nu(t)}\right) = \Phi_S^\omega(\mathbf{x})\Phi_T^\nu(\mathbf{x})$$

■

Example: Compositions

$S = \text{all compositions} = \bigcup_{k=0}^{\infty}(\mathbb{N}_{\geq 1})^k$

Define $\phi((c_1, c_2, \dots, c_k)) = (c_1 + \cdots + c_k, k)$ (sum of parts, # of parts)
We use the following weight function on $\mathbb{N}_{\geq 1}$
Define $\omega: \mathbb{N}_{\geq 1} \to \mathbb{N}^2,\ \omega(c) = (c, 1)$
Note that $\phi((c_1, \dots, c_k)) = (c_1 + \cdots + c_k, k)$
$\omega(c_1) + \cdots + \omega(c_k) = (c_1 + \cdots + c_k, k)$
So product lemma applies.

By Sum Lemma

$$\Phi_S^\phi(x,y) = \sum_{k=0}^{\infty}\Phi_{(\mathbb{N}_{\geq 1})^k}^\phi(x,y)$$

By Product Lemma

$$\Phi_{(\mathbb{N}_{\geq 1})^k}^\phi(x,y) = \left(\Phi_{\mathbb{N}_{\geq 1}}^\omega(x,y)\right)^k = \left(\frac{yx}{1-x}\right)^k$$

$$\therefore \Phi_S^\phi(x,y) = \sum_{k=0}^{\infty}\left(\frac{yx}{1-x}\right)^k = \frac{1}{1-\left(\frac{yx}{1-x}\right)}$$

Answer:

$$[x^ny^k]\frac{1}{1-\left(\frac{yx}{1-x}\right)}$$

is # of compositions of n with k parts.

Strings

October-02-13 10:32 AM

Finite Strings

Let S be a set with weight function $\omega: S \to \mathbb{N}^r$
We define the set of all finite strings on S to be

$$S^* \coloneqq \bigcup_{k=0}^{\infty} S^k$$

If $\sigma \in S^*$, then $\sigma \in S^k$ for some unique $k \in \mathbb{N}$. If $k = 0$, then $\sigma = \epsilon$ is the **empty string**. k is called the **length** of σ, denoted $l(\sigma)$.

We define a weight function on S^*, $\omega^*: S^* \to \mathbb{N}^r$
If $\sigma = (s_1, \dots, s_k)$ then $\omega^*(\sigma) = \omega(s_1) + \dots + \omega(s_k)$

Lemma

If S is a set with weight function $\omega: S \to \mathbb{N}^r$, then ω^* is a weight function on S^* if and only if $\omega(s) \neq \mathbf{0}\ \forall s \in S$

Proposition (Finite String Lemma)

Let S be a set with weight funciton $\omega: S \to \mathbb{N}^r$ such that $\omega(s) \neq \mathbf{0}\ \forall s \in S$.
Then $\Phi_{S^*}^{\omega^*}(\mathbf{x}) = \dfrac{1}{1 - \Phi_S^{\omega}(\mathbf{x})}$

Proof of Lemma

Suppose $\omega(z) = \mathbf{0}$ for some $z \in S$. Then consider $z^k = (z, z, \dots, z)$.
By definition, $\omega^*(z^k) = \omega(z) + \dots + \omega(z) = k\omega(z) = \mathbf{0}$.
$\therefore z^k \in (\omega^*)^{-1}(\mathbf{0})\ \forall k$
$\Rightarrow |(\omega^*)^{-1}(\mathbf{0})|$ is infinite
$\therefore \omega^*$ is not a weight function.

Suppose $\omega(s) \neq \mathbf{0}\ \ \forall s \in S$
For any $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$, let $|\beta| = \beta_1 + \beta_2 + \dots + \beta_r$
Since $\omega(s) \neq \mathbf{0}$, $|\omega(\sigma)| \geq 1\ \ \forall s \in S$
$\therefore$ If $\sigma = (s_1, \dots, s_k)$ then $|\omega^*(\sigma)| = |\omega(s_1)| + |\omega(s_2)| + \dots + |\omega(s_k)| \geq k$
$\therefore |\omega^*(\sigma)| \geq l(\sigma)\ \ \forall \sigma \in S^*$
Let $\alpha \in \mathbb{N}^r$. We want to show that $(\omega^*)^{-1}(\alpha)$ is finite.
If $\omega^*(\sigma) = \alpha$ and $\sigma = (s_1, \dots, s_k)$ then $\omega(s_i) \leq \alpha$.
That is, if we define $U_\alpha \coloneqq \bigcup_{\beta \leq \alpha} \omega^{-1}(\beta)$ then $s_i \in U_\alpha\ \ \forall i = 1, \dots, k$
Note that there are finitely many $\beta \leq \alpha$ and $\omega^{-1}(\beta)$ is finite, so U_α is a finite union of finite sets. Hence U_α is finite.
Since $l(\sigma) \leq |\omega(\sigma)| = |\alpha|$
Then $\sigma \in U_\alpha^0 \cup U_\alpha^1 \cup \dots \cup U_\alpha^{|\alpha|}$, which is also a finite union of finite sets
Thus we have shown that $\omega^{-1}(\alpha) \subseteq \bigcup_{i=0}^{|\alpha|} U_\alpha^i$
$\therefore\ \omega^{-1}(\alpha)$ is finite, as required
■

Finite String Lemma

- [ ] Homework

Example

The set of compositions is $(\mathbb{N}_{\geq 1})^*$
Define weight function $\omega: \mathbb{N}_{\geq 1} \to \mathbb{N}^2$, $\omega(c) = (c, 1)$
Then $\omega^*(c_1, \dots, c_k) = (c_1 + \dots + c_k, k)$

$$\Phi_{\mathbb{N}_{\geq 1}^*}^{\omega^*}(x, y) = \frac{1}{1 - \Phi_{\mathbb{N}_{\geq 1}}^{\omega}(x, y)} = \frac{1}{1 - \frac{xy}{1 - x}}$$

The q-Binomial Theorem

October-02-13 11:02 AM

Inversion

Let $\sigma \in S_n$, $\sigma = a_1 a_2 \dots a_n$ be a permutation of length n. An **inverstion** of σ is a pair (i, j) with $1 \le i < j \le n$ and $a_i > a_j$. Let $\text{inv}(\sigma) = \#$ of inversions of σ.

Note

For $\sigma \in S_n$, $0 \le \text{inv}(\sigma) \le \binom{n}{2}$

Notation

For $k \in \mathbb{N}$, $[k]_q$ is the following polynomial of q:

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1} = \sum_{i=0}^{k-1} q^i = \frac{1 - q^k}{1 - q}$$

For $n \in \mathbb{N}$, $[n]!_q = [n]_q [n-1]_q [n-2]_q \cdots [2]_q [1]_q$ where $[0]!_q = 1$

Note

$[n]!_q$ is a polynomial in q of degree $(n-1) + (n-2) + \dots + 2 + 1 + 0 = \frac{n(n-1)}{2} = \binom{n}{2}$

Theorem

$$\Phi_{S_n}^{\text{inv}}(q) = \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = [n]!_q$$

Theorem

$$B_{n,k}(q) = \sum_{A \in \mathcal{B}(n,k)} q^{\text{sum}(A)} = q^{\frac{k(k+1)}{2}} \frac{[n]!_q}{[k]\,!_q \cdot [n-k]\,!_q}$$

Note

Plugin in $q = 1$ RHS $\to 1^{\frac{k(k+1)}{2}} \cdot \frac{n!}{k!(n-k)!} = \binom{n}{k}$

Define $\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q}$

This is called the q −binomial coefficient. Also called Gaussian polynomials.

q-Binomial Theorem

$$(1 + qx)(1 + q^2 x) \cdots (1 + q^n x) = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \binom{n}{k}_q x^k$$

Polynomial Degrees

$[n]_q$ plug in $q = 1 \Rightarrow n$, degree $n - 1$

$[n]!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q$ plug in $q = 1 \Rightarrow n!$, degree $\binom{n}{2}$

$\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$ plug in $q = 1 \Rightarrow \binom{n}{k}$,

degree $\binom{n}{2} - \binom{k}{2} - \binom{n-k}{2} = k(n-k)$

Theorem 3

$$\Phi_{\mathcal{B}(n,k)}^{\text{sum}}(q) = q^{\frac{k(k+1)}{2}} \binom{n}{k}_q$$

Interpretations

There are other interpretations of $\binom{n}{k}_q$:

(Stated without proof)

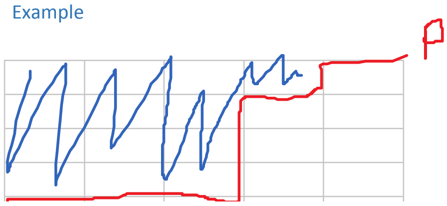
Theorem

Let $\mathcal{L}(a, b)$ be the set of lattice paths from $(0,0)$ to (a, b). Define the weight function area: $\mathcal{L}(a, b) \to \mathbb{N}$

area(P) = the area bounded by P and the path $(0,0) \to (0, b) \to (a, b)$

Then $\Phi_{\mathcal{L}(a,b)}^{\text{area}}(q) = \binom{a+b}{b}_q$

Example



Recall $\mathcal{P}(X)$ = set of all subsets of X

Binomial Theorem

Let $\omega: \mathcal{P}(N_n) \to \mathbb{N}$, $\omega(S) = |S|$

What is $\Phi_{\mathcal{P}(N_n)}^{\omega}(\mathbf{x})$?

Method 1

$[\mathbf{x}^k] = \Phi_{\mathcal{P}(N_n)}^{\omega}(\mathbf{x}) = |\omega^{-1}(k)| = |\mathcal{B}(n,k)| = \binom{n}{k}$

$$\therefore \Phi_{\mathcal{P}(N_n)}^{\omega}(\mathbf{x}) = \sum_{k=0}^{n} \binom{n}{k} \mathbf{x}^k$$

Method 2

We have a bijection $F: \mathcal{P}(N_n) \to \{0,1\}^n$

$F(S) = (a_1, \dots, a_n)$ where $a_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$

We have a weight function on $\mathcal{P}(N_n)$. Want to put one on $\{0,1\}^n$ to make this a weight preserving bijection.

$\nu: \{0,1\}^n \to \mathbb{N}$, $\nu(a_1, \dots, a_n) = \omega(F^{-1}(a_1, \dots, a_n)) = a_1 + \dots + a_n$

$$\Phi_{\mathcal{P}(N_n)}^{\omega}(\mathbf{x}) = \Phi_{\{0,1\}^n}^{\nu}(\mathbf{x}) = \left(\Phi_{\{0,1\}}^{(?)}(\mathbf{x})\right)^n = (1 + \mathbf{x})^n$$

Putting these together

$$(1 + \mathbf{x})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{x}^k$$

Generalization

Now, change the weight function

$\omega: \mathcal{P}(N_n) \to \mathbb{N}^2$, $\omega(S) = (|S|, \text{sum}(S))$, where $\text{sum}(S) = \sum_{s \in S} s$

Example

$n = 7$, $S = \{1,3,6\}$

$\omega(s) = (3, 10)$

Method 2 Generalization

We have a bijection $F: \mathcal{P}(N_n) \to \{0,1\}^n$

$F(S) = (a_1, \dots, a_n)$ where $a_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$

We have a weight function on $\mathcal{P}(N_n)$. Want to put one on $\{0,1\}^n$ to make this a weight preserving bijection.

$\nu: \{0,1\}^n \to \mathbb{N}^2$

$\nu(a_1, \dots, a_n) = \omega(F^{-1}(a_1, \dots, a_n)) = (a_1 + a_2 + \dots + a_n,\quad a_1 + 2a_2 + 3a_3 + \dots + na_n)$

$$= \left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} i a_i\right)$$

Define $\mu_i: \{0,1\} \to \mathbb{N}^2$, $\mu_i(a) = (a, ia)$

Then $\nu(a_1, \dots, a_n) = \sum_{i=1}^{n} \mu_i(a_i)$

$\therefore \Phi_{\mathcal{P}(N_n)}^{\omega}(x, q) = \Phi_{\{0,1\}^n}^{\nu}(x, q) = \Phi_{\{0,1\}}^{\mu_1}(x, q) \Phi_{\{0,1\}}^{\mu_2}(x, q) \cdots \Phi_{\{0,1\}}^{\mu_n}(x, q)$

$= (1 + qx)(1 + q^2 x) \cdots (1 + q^n x)$

Method 1 Generalization

$$\Phi_{\mathcal{P}(N_n)}^{\omega}(x, q) = \sum_{S \subseteq N_n} x^{|S|} q^{\text{sum}(S)} = \sum_{k=0}^{n} \left(\sum_{S \subseteq \mathcal{B}(n,k)} q^{\text{sum}(S)} \right) x^k$$

Let $B_{n,k} = \sum_{S \in \mathcal{B}(n,k)} q^{\text{sum}(S)}$

$$\therefore \Phi_{\mathcal{P}(N_n)}^{\omega}(x, q) = \sum_{k=0}^{n} B_{n,k}(q) x^k$$

If we can figure out what $B_{n,k}(q)$ is then we'll get an analogue fo the binomial theorem.

Note $\Phi_{\mathcal{B}(n,k)}^{\text{sum}}(q) = \sum_{S \in \mathcal{B}(n,k)} q^{\text{sum}(S)} = B_{n,k}(q)$

Plug in $q = 1$, $B_{n,k}(q) = \sum_{S \in \mathcal{B}(n,k)} 1 = |\mathcal{B}(n,k)| = \binom{n}{k}$

Idea: Turn material from chapter 2 into weight preserving bijections.
Start with permutations.

Example of Inversions

$\sigma = 5\ 1\ 4\ 2\ 6\ 3$

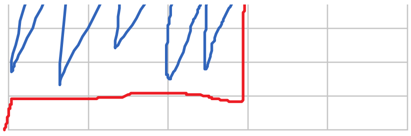
The inversions of σ are

$(1,2), (1,3), (1,4), (1,6), (3,1), (3,6), (5,6)$

so $\text{inv}(\sigma) = 7$

Note: Inversions are pairs of indices, not pairs of elements.

Example



$\text{area}(P) = 13$

Theorem

Fix $0 \le k \le n$ integers. Let $q = p^c$ be a prime power. Let $\mathbb{F}$ be the field with q elements.
Let V be an n-dimensional vector spece over $\mathbb{F}$.
Then the number of k-dimensional linear subspaces of V is $\binom{n}{k}_q$

Theorem

Let V be an infinite dimensional vector space over $\mathbb{R}$. Let $q \in \mathbb{R}$
Suppose $X, Y \in \mathcal{L}(V)$ satisfying $YX = qXY$

$$(X+Y)^n = \sum_{k=1}^{n} \binom{n}{k}_q X^k Y^{n-k}$$

Example YX=qXY

For $a = (a_n)_{n\in\mathbb{N}}$, let $X_a = \big((x_a)_n\big)_{n\in\mathbb{N}} = (q^n a_n)_{n\in\mathbb{N}}$
$(a_0, a_1, a_2, \dots) \mapsto (a_0, qa_1, q^2 a_2, \dots)$
Let $Y_a = \big((y_a)_n\big)_{n\in\mathbb{N}} = (a_{n+1})_{n\in\mathbb{N}}$
$(a_0, a_1, a_2, \dots) \mapsto (a_1, a_2, a_3, \dots)$
Then $YX = qXY$

so $\text{inv}(\sigma) = 7$
Note: Inversions are pairs of indices, not pairs of elements.

Example

$\sigma = 1\,2\,3\,4\,5\,6$ has no inversions. $\text{inv}(\sigma) = 0$, the minimum possible

Example

$\sigma = 6\,5\,4\,3\,2\,1$ has all possible inversions. $\text{inv}(\sigma) = 15 = \binom{6}{2}$

Proof of Theorem

Recall that we had $S_n \rightleftarrows Q_n = N_n \times N_{n-1} \times \cdots \times N_1$
with maps $I_n: S_n \to q_n,\ J_n: Q_n \to S_n$

Claim

If $\sigma \in S_n, I_n(\sigma) = (1 + r_1, \dots, 1 + r_n)$ then $\text{inv}(\sigma) = r_1 + r_2 + \cdots + r_n$
To see this, recall that $r_i = \big|\{j \mid i \le j \le n \text{ and } a_i > a_j\}\big| = \#$ of inversions of the form $(i, *)$ where i is fixed.
Summing over all i gives the total number of inversions.

Now define weight functions $\mu_i: N_{n+1-i} \Rightarrow \mathbb{N},\ \mu_i(h) = (h-1)$
$\nu: Q_n \to \mathbb{N}$,
$\nu(h_1, \dots, h_2) = (h_1 - 1) + (h_2 - 2) + \cdots + (h_n - 1)$
$= \mu_1(h_1) + \mu_2(h_2) + \cdots + \mu_n(h_n)$

We just showed $I_n: S_n \to Q_n$ is a wight perserving bijection with these weight functions.
$\Phi_{S_n}^{\text{inv}}(q) = \Phi_{Q_n}^{\nu}(q) = \Phi_{N_n}^{\mu_1}(q)\Phi_{N_{n-1}}^{\mu_2}(q)\cdots\Phi_1^{\mu_n}(q)$
$= (1 + q + \cdots + q^{n-1})(1 + q + \cdots + q^{n-2}) \cdots (1+q)(1) = [n]_q[n-1]_q \cdots [2]_q[1]_q$
$= [n]!_q$

Proof of Theorem 3

Recall that we had
$S_n \rightleftarrows \mathcal{B}(n,k) \times S_k \times S_{n-k}$
given by the maps
$\Psi_{n,k}: S_n \to \mathcal{B}(n,k) \times S_k \times S_{n-k}$
$\Phi_{n,k}: \mathcal{B}(n,k) \times S_k \times S_{n-k} \to S_n$

Claim

$\Psi_{n,k}(\sigma) = (A, \beta, \gamma)$ then

$$\text{inv}(\sigma) = \left[\text{sum}(A) - \frac{k(k+1)}{2}\right] + \text{inv}(\beta) + \text{inv}(\gamma)$$

To see this, split the sets of inversions of σ into three sets:
$E_1 = \{\text{inversions of } \sigma\ (i,j) : i < j \le k\}$
$E_2 = \{\text{inversions of } \sigma\ (i,j) : k+1 \le i < j\}$
$E_3 = \{\text{inversions of } \sigma\ (i,j) : i \le k, j \ge k+1\}$
Then $\text{inv}(\sigma) = |E_1| + |E_2| + |E_3|$
$|E_1| = \text{inv}(\beta)$

Example

$n = 7, \quad k = 3, \quad \sigma = 5\,1\,4\,2\,6\,7\,3$
$A = \{1,2,4,5\}, \quad \beta = 4\,1\,3\,2, \quad \gamma = 2\,3\,1$
E_1 is the set of inversions of β
Similarly, $|E_2| = \text{inv}(\beta)$

Finally, $|E_3| = \text{sum}(A) - \frac{k(k+1)}{2}$
Write $\sigma = a_1 a_2 \cdots a_n$
$(i,j) \in E_3$ iff $(a_i, a_j) \in N_n \times N_n$ is a pair such that $a_i \in A, a_j \notin A$ and $a_i > a_j$
How many pairs $(a, z) \in N_n \times N_n$ are there with $a \in A, z \in N_n \backslash A$ and $a > z$?
Sort the elements of A as $s_1 < s_2 < \cdots < s_k$
For each s_i, there are $i - 1$ elements of A that are smaller than s_i. And there are $s_i - 1$ elements of N_n that are smaller than i.
$\therefore$ there are $(s_i - 1) - (i - 1) = s_i - i$ elements of $N_n \setminus A$ that are smaller than s_i
$\therefore$ total $\# = (s_1 - 1) + (s_2 - 2) + \cdots + (s_k - k) = \text{sum}(A) - \dfrac{k(k+1)}{2}$
This proves the claim

This shows that
$\Phi_{S_n}^{\text{inv}}(q) = \Phi_{\mathcal{B}(n,k)}^{\nu}(q) \cdot \Phi_{S_k}^{\text{inv}}(q) \cdot \Phi_{S_{n-k}}^{\text{inv}}(q)$
where $\nu(A) = \text{sum}(A) - \frac{k(k+1)}{2}$
Note $\Phi_{\mathcal{B}(n,k)}^{\nu}(A) = q^{-\frac{k(k+1)}{2}} \Phi_{\mathcal{B}(n,k)}^{\text{sum}}(q)$

$[n]!_q = q^{-\frac{k(k+1)}{2}} \Phi_{\mathcal{B}(n,k)}^{\text{sum}}(q) [k]!_q\, [n-k]!_q$

Rearranging:

$$\Phi_{\mathcal{B}(n,k)}^{\text{sum}}(q) = q^{\frac{k(k+1)}{2}} \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

■

$$\therefore \Phi^{\omega}_{\mathcal{P}(N_n)} = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \binom{n}{k}_q x^k$$

where $\omega(S) = \big(|S|, \mathrm{sum}(S)\big)$
Putting the two methods to get

q-Binomial Theorem

For $n \in \mathbb{N}$

$$(1+qx)(1+q^2x)\cdots(1+q^nx) = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \binom{n}{k} x^k$$

Recursive Structures

October-09-13 10:37 AM

General Binomial Theorem

For $\alpha \in \mathbb{C},\ x \in \mathbb{C},\ |x| < 1$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Where $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$

Note: $\ln(x)$ is not well defined for $x \in \mathbb{C}$

Can also use this if x is a formal indeterminant.

Proposition

a) $\sqrt{1-4x} = 1 - 2\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$

b) $\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$

Binary Rooted Tree (BRT)

A binary rooted tree is a tree with a distinguished node $\odot$ called the root, in which every node has at most two children, one called left (child) and the other right. If there is only one child, it is either left or right.
A **terminal** is a node with no children. (Similar to a leaf, but not exactly the same.

Let $\mathcal{T}$ be the set of all BRTs. For $T \in \mathcal{T}$, let
$n(T)$ = number of nodes
$\tau(T)$ = number of terminals

Catalan Numbers

The numbers $\frac{1}{n+1}\binom{2n}{n}$ are called the Catalan numbers.

Method for Solving Generating Function Problems

1. Identify the set of objects under consideration (remove parameters). Reintroduce these parameters as a weight function.
2. Describe S in a more formal way / find a bijection involving S.
3. Define weight functions such that
   - bijections are weight preserving
   - weight of a composite object = sum of weights of pieces.

   If weight functions are already defined check that this is true.
4. Use sum and product lemmas, etc. to get an equation for generating function of S.
5. Extract required information from the generating function.

Ch. 6: Recursive Structures

Proposition

a) $\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-4x)^k = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-1)^k 4^k x^k$

Now, if $k \geq 1$

$$\binom{\frac{1}{2}}{k}(-1)^k 4^k = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}(-1)^k \cdot 4^k$$

$$= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!}(-1)^k 2^k 2^k = -\left[\frac{1\cdot 1\cdot 3\cdot 5\cdots(2k-3)}{k!}\right]2^k$$

$$= -\left[\frac{1\cdot 3\cdot 5\cdots(2k-3)\cdot 2\cdot 4\cdots(2k-2)}{k!\cdot 2\cdot 4\cdot 6\cdots(2k-2)}\right]2^k = -\frac{1}{2}\cdot\frac{(2k-2)!}{k!\cdot 1\cdot 2\cdot 3\cdots(k-1)}$$

$$= \frac{2}{k}\left(\frac{(2k-2)!}{(k-1)!\,(k-1)!}\right) = -\frac{2}{k}\binom{2k-2}{k-1}$$

If $k=0$, $\binom{\frac{1}{2}}{k}(-1)^k 4^k = 1$

Continuing,

$$\sqrt{1-4x} = \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-1)^k 4^k x^k = 1 + \sum_{k=1}^{\infty} -\frac{2}{k}\binom{2k-2}{k-1}x^k = 1 - 2\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}$$

b) Similar. ■

Question

Given a random BRT with n nodes, what is the expected number of terminals?
Equivalently, what is the average number of terminals among all BRTs with n nodes.
Random = chosen uniform at random from the set of all BRTs with n nodes.

$\omega(T) = (\lambda(T), \tau(T))$
To answer our question: we'll compute $\Phi_{\mathcal{T}}^{\omega}(x,y)$ and get answer from here.
As a warm up, let's compute $\Phi_{\mathcal{T}}^{n}(x)$, weight function is # of nodes.

Describe $\mathcal{T}$ in another way. Given a BRT $T \in \mathcal{T}$, remove the root node $\odot$ to obtain a pair of "subtrees" (L,R) which may be either BRTs or empty.
L is rooted at the left child of $\odot$
R is rooted at the right child off $\odot$
This construction gives a bijection $\mathcal{T} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\emptyset\}) \times (\mathcal{T} \cup \{\emptyset\}),\ T \mapsto (\odot, L, R)$
Note that if $T \leftrightarrow (\odot, L, R),\ n(T) = n(\odot) + n(L) + n(R) = 1 + n(L) + n(R)$
So we can use the product lemma:
$\Phi_{\mathcal{T}}^{n}(x) = \Phi_{\{\odot\}}^{n}(x)\Phi_{\mathcal{T}\cup\{\emptyset\}}^{n}(x)\Phi_{\mathcal{T}\cup\{\emptyset\}}^{n}(x) = x(\Phi_{\mathcal{T}}^{n}(x)+1)^2$
Let $A = A(x) = \Phi_{\mathcal{T}}^{n}(x)$
$A = x(A+1)^2 = xA^2 + 2xA + x$
$xA^2 + (2x-1)A + x = 0$

$$A = \frac{-(2x-1) \pm \sqrt{(2x-1)^2 - 4x^2}}{2x} = -1 + \frac{1}{2x} \pm \frac{1}{2x}\sqrt{1-4x}$$

$$= -1 + \frac{1}{2x} \pm \frac{1}{2x}\left(1 - 2\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}\right)$$

Need "-" solution or else this won't be a power series. The "+" solution has a $\frac{1}{x}$ term.

$$A(x) = -1 + \frac{1}{2x} - \frac{1}{2x}\left(1 - 2\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}\right) = -1 + \frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}$$

$$= -1 + \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n = \sum_{n=1}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

∴ #BRTs with n nodes is

$$[x^n]\Phi_{\mathcal{T}}^{n}(x) = [x^n]A(x) = \begin{cases} \frac{1}{n+1}\binom{2n}{n} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

Now let's try to do this with ω.
We have our bijection
$\mathcal{T} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\emptyset\})^2, \qquad T \leftrightarrow (\odot, L, R)$
$\omega(T) = (n(T), \tau(T))$
$n(T) = 1 + n(L) + n(R)$

$$\tau(T) = \begin{cases} \tau(L) + \tau(R) & \text{if } L \neq \emptyset \text{ or } R \neq \emptyset \\ 1 & \text{if } L = R = \emptyset \end{cases}$$

It is not true that $\omega(T) = \omega(\odot) + \omega(L) + \omega(R)$ so we need to fix up the weight function.
Don't' want to modify $\omega(T)$ since that is how we get our answer.

Define weight function $\mu: \{\odot\} \to \mathbb{N}^2,\ \mu(\odot) = (1,0)$
then if $L \neq \emptyset$ or $R \neq \emptyset, \omega(T) = \mu(\odot) + \omega(L) + \omega(R)$
This does not hold if $L = R = \emptyset$ so we remove that case from our sets.

$\therefore$ we have a weight preserving bijection
$\mathcal{T} \setminus \{\odot\} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\odot\})^2 \setminus \{(\odot, \emptyset, \emptyset)\}$
$\Phi^{\omega}_{\mathcal{T}}(x,y) - \Phi^{\omega}_{\{\odot\}}(x,y) = \Phi^{\mu}_{\{\odot\}}(x,y)\left(\Phi^{\omega}_{\mathcal{T}\cup\{\emptyset\}}(x,y)\right)^2 - \Phi^{\mu\oplus\omega\oplus\omega}_{\{(\odot,\emptyset,\emptyset)\}}(x,y)$
Let $A = A(x,y) = \Phi^{\omega}_{\mathcal{T}}(x,y)$
$A - xy = x(A+1)^2 - x$
$A = x(1+A)^2 - x + xy, \qquad -x + xy$ is a correction for the root behaviour

Solve this with quadratic equation
$$A = \frac{1 - 2x \pm \sqrt{(1-2x)^2 - 4x^2 y}}{2x}$$
Note: if we put $y = 1$ we're supposed to get
$$\Phi^{\omega}_{\mathcal{T}}(x,1) = \sum_{T\in\mathcal{T}} x^{n(T)} 1^{\tau(T)} = \sum_{T\in\mathcal{T}} x^{n(T)} = \Phi^{n}_{\mathcal{T}}(x)$$
So we need the "-" solution
$$A(x,y) = \frac{1 - 2x \pm \sqrt{(1-2x)^2 - 4x^2 y}}{2x}$$
We want to know the expected # of terminals in a tree with n nodes.
$$\frac{\sum_{\substack{T\in\mathcal{T}\\ n(T)=n}} \tau(T)}{|\{T \in \mathcal{T} : n(T) = n\}|}$$

$$A(x,y) = \sum_{T\in\mathcal{T}} x^{n(T)} y^{\tau(T)} = \sum_{n=0}^{\infty}\left(\sum_{\substack{T\in\mathcal{T}\\ n(T)=n}} y^{\tau(T)}\right) x^n$$
$$\frac{\partial}{\partial y} A(x,y) = \sum_{n=0}^{\infty}\left(\sum_{\substack{T\in\mathcal{T}\\ n(T)=n}} \tau(T) y^{\tau(T)-1}\right) x^n$$
Plug in $y = 1$
$$\left.\frac{\partial}{\partial y} A(x,y)\right|_{y=1} = \sum_{n=0}^{\infty}\left(\sum_{\substack{T\in\mathcal{T}\\ n(T)=n}} \tau(T)\right) x^n$$
$$[x^n] \left.\frac{\partial}{\partial y} A(x,y)\right|_{y=1} = \text{numerator}$$
In our case,
$$A(x,y) = \frac{1 - 2x - \sqrt{(1-2x)^2 - 4x^2 y}}{2x}$$
$$\frac{\partial}{\partial y} A(x,y) = \frac{x}{\sqrt{(1-2x)^2 - 4x^2 y}}$$
$$\left.\frac{\partial}{\partial y} A(x,y)\right|_{y=1} = \frac{x}{\sqrt{1-4x}} = x\sum_{n=0}^{\infty}\binom{2n}{n} x^n$$
$$[x^n] \left.\frac{\partial}{\partial y} A(x,y)\right|_{y=1} = \binom{2n-2}{n-1}$$
Final answer:
$$\frac{\binom{2n-2}{n-1}}{\frac{1}{n+1}\binom{2n}{n}}$$

More Recursive Structures

October-21-13 10:31 AM

Plane Planted Tree

A **plane planted tree** is a tree with a designated node $\odot$ called the root, drawn in the plane such that each node has its children ordered from left to right.

Rooted trees with implicit node labels.

Super-Diagonal Lattice Paths

a.k.a. Dyck Paths, Catalan Paths

Lattice paths that are always above the diagonal line.

There are $\frac{1}{n+1}\binom{2n}{n}$ SDLPs from $(0,0)$ to (n,n)

Why (Partial) Derivatives?

- IT WORKS
- Derivative has a combinatorial meaning: **marking / rooting**

Example

Let $S=\{0,1\}^*$, $\omega: S\to\mathbb{N}^2$, $\omega(\sigma)=(\#\text{ of }0s, \#\text{ of }1s)$

$$\Phi_S^{\omega}(x,y)=\frac{1}{1-\Phi_{\{0,1\}}(x,y)}=\frac{1}{1-(x+y)}$$

Let $\mathcal{T}$ be the set of all {0,1}-strings in which exactly one of the 1s is marked (circled).

$\mathcal{T}=\{\boxed{1}, 0\boxed{1}, \boxed{1}0, \boxed{1}1, 1\boxed{1}, 00\boxed{1}, \dots, \boxed{1}11, 1\boxed{1}1, 11\boxed{1}, \dots\}$

Weight function: $\phi(\sigma)=(\#\text{ of }0s, \#\text{ of }1s)$

Relationship

$$y\frac{\partial}{\partial y}\Phi_S^{\omega}(x,y)=\Phi_{\mathcal{T}}^{\phi}(x,y)$$

- [ ] Check this

We'll see this again in Exponential Generating functions.

Example (Plane Planted Trees)

How many PPTs with n nodes?
Let $\mathcal{U}$ be the set of all PPTs.
Let $n(T)=\#$ of nodes for $T\in\mathcal{U}$
Want to compute $\Phi_{\mathcal{U}}^{n}(x)$

Bijection: If we remove the root of a PPT we get a tuple of PPTs
$\mathcal{U}\rightleftarrows\{\odot\}\times\mathcal{U}^*$
$T\leftrightarrow(\odot, c_1, c_2, \dots, c_k)$
$n(T)=n(\odot)+n(c_1)+n(c_2)+\cdots+n(c_k)$

$$\therefore\ \Phi_{\mathcal{U}}^{n}(x)=\Phi_{\{\odot\}}^{n}(x)\Phi_{\mathcal{U}^*}^{n}(x)=\Phi_{\{\odot\}}^{n}(x)\frac{1}{1-\Phi_{\mathcal{U}^*}^{n}(x)}=\frac{x}{1-\Phi_{\mathcal{U}^*}^{n}(x)}$$

$$A=\frac{x}{1-A},\qquad A^2-A+x=0$$

$$A=\frac{1\pm\sqrt{1-4x}}{2}=\frac{1}{2}\pm\frac{1}{2}\left(1-2\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}\right)$$

Since we know $[x^0]A(x)=\#$ of PPTs with 0 nodes $=0$ it must be the "-" solution.

$$\therefore A(x)=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}$$

$\therefore$ The # of PPTs with $(n+1)$ nodes is $\frac{1}{n+1}\binom{2n}{n}$, $n\geq 1$

Bijection between BRTs with n nodes and PPTs with n+1 nodes

Starting with BRT → create a left child for every node missing one → contract all edges to right children → have a PPT

Super-Diagonal Lattice Path Bijections

$X=$ all SDLPs
$X\rightleftarrows(\{N\}\times X\times\{E\})^*$

Formal Power Series

October-23-13 10:33 AM

Commutative Ring

Algebraic operations: $+, \cdot$, -
Special Elements: 0 (additive identity), 1 (multiplicative identity)

$+$ and $\cdot$ are associative, commutative, distributive
$-$ is inverse of $+$

Inverses

If R is a ring, $a, b \in R$ and $ab = 1$ then a and b are **invertible**, b is the **inverse** of a, $b = a^{-1}$

Field

If every non-zero element of F has an inverse, R is a field.
(Also require $0 \neq 1$)

Zero-Divisors

$a, b \in R, \quad a \neq 0, b \neq 0$
If $ab = 0$ then we say a and b are zero-divisors.
- [ ] A zero-divisor can't be invertible.

Integral Domain

A ring with no zero-divisors is called an Integral Domain.

Mostly we'll want to have integral domains (or even fields).

Example Rings

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
$\mathbb{Z}_n$
$\mathcal{F}(\mathbb{R}, \mathbb{R}) =$ functions $\mathbb{R} \to \mathbb{R}$

Example Integral Domain

$\mathbb{Z}$ is an integral domain
$\mathbb{Z}_{15}$ is not an integral domain since $[5] \cdot [3] = [0]$
$\mathbb{Z}_n$ is an integral domain iff n is prime
$\mathcal{F}(\mathbb{R}, \mathbb{R})$ is not ain integral domain.

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

$f(x)g(x) = 0 \therefore f$ and g are zero divisors
Every field is an integral domain

Why are integral domains good?

Integral domains have a weak form of "division"
If $a, b \neq 0$, $a = bc$, Write $\frac{a}{b} = c$

Example

$\frac{6}{3} = 2$ but $\frac{5}{3}$ is not defined in $\mathbb{Z}$

This does not work well if R is not an integral domain
For example, in $\mathbb{Z}_{15}$

$\frac{[6]}{[3]} = [2]$ because $[6] = [3][2]$
$\frac{[6]}{[3]} = [7]$ because $[3][7] = [6]$

Division is not well defined.
- [x] Division is well defined in an integral domain

Suppose $ab = ac \Rightarrow ab - ac = 0 \Rightarrow a(b - c) = 0$
$\Rightarrow a = 0$ or $b = c$
If $a = 0$ then $ab = 0$ so $\frac{0}{b} = \frac{0}{c} = 0$

Constructions

1. **Ring of Polynomials**
   Let R be a ring
   $R[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \in \mathbb{N}, \quad a_0, a_1, \ldots, a_n \in R\}$
   $+, -, \cdot$ defined
   1, 0 are constant polynomials
   x is an indeterminant

   $R[x]$ is always a commutative ring.
   If $R[x]$ is an integral domain if R is.
   $R[x]$ is never a field.
   For example, x does not have an inverse in $R[x]$
   Polynomials can be evaluated - can plugin values in R for x

2. **Ring of Rational Functions**
   Let R be an integral domain
   $R(x) = \left\{ \frac{f(x)}{g(x)} \mid f(x), g(x) \in R[x], \quad g \neq 0 \right\}$
   Subject to usual simple function rules.
   Addition
   $$\frac{f(x)}{g(x)} + \frac{h(x)}{k(x)} = \frac{k(x)f(x) + h(x)g(x)}{g(x)k(x)}$$
   Multiplication
   $$\frac{f(x)}{g(x)} \cdot \frac{h(x)}{k(x)} = \frac{f(x)h(x)}{g(x)k(x)}$$
   $R(x)$ is a field (The inverse of $\frac{f(x)}{g(x)}$ is $\frac{g(x)}{f(x)}$)
   Evaluation: Sort of. If $a \in R$, $g(a) \neq 0$ we can maybe make sense of $\frac{f(a)}{g(a)}$

3. **Ring of Formal Power Series**
   R is a ring
   $R[[x]] = \{f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \mid a_n \in R \ \forall n \in \mathbb{N}\}$
   $+, \cdot$ defined in the "obvious way"
   - Collect all like terms
   - Use distributive law

   $$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n$$
   $$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$f(x)g(x) = \left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i b_j x^{i+j} = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n$$

$0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$

$1 = 1 + 0x + 0x^2 + 0x^3 + \cdots$

More Formal Power Series

October-25-13 10:33 AM

Theorem (Inverse of Formal Power Series)

$$f(x)=\sum_{n=0}^{\infty} a_n x^n \in R[[x]]$$

is invertible in $R[[x]]$ if and only if a_0 is invertible in R.

Special Case

If R is a field, then $f(x)$ is invertible if and only if $a_0 \neq 0$.

Index of a Formal Laurent Series

The index of $f(x) \in R((x))$ is the smallest number $n \in \mathbb{Z}$ such that $[x^n]f(x) \neq 0$

Denoted $I(f)$

Convention: $I(0)=+\infty$

Inverse of Formal Power Series

$f(x)=\sum_{n=0}^{\infty} a_n x^n$, when is $f(x)$ invertible?

Suppose $f(x)g(x)=1,\ g(x)=\sum_{n=0}^{\infty} b_n x^n$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_i b_{n-i}\right)x^n = 1+0x+0x^2+\cdots$$

Comparing coefficients

$[x^0]$: $a_0 b_0 = 1 \Rightarrow a_0$ must be invertible and $b_0=a_0^{-1}$

$[x^1]$: $a_0b_1+a_1b_0=0 \Rightarrow b_1=-a_0^{-1}a_1b_0=-a_0^{-1}a_1a_0^{-1}$

$[x^2]$: $a_0b_2+a_1b_1+a_2b_0=0 \Rightarrow\ b_2=-a_0^{-1}(a_1b_1+a_2b_0)$

$$b_n=-a_0^{-1}\left(\sum_{i=1}^{n} a_i b_{n-i}\right)$$

Once a_0 is invertible, we can solve for $b_0, b_1, b_2, \ldots$

Exercise

- $R[[x]]$ is an integral domain if R is an integral domain
- $R[[x]]$ is never a field if $x \in R[[x]]$ is never invertible.

Constructions (Continued)

4. $R((x))$ Formal Laurent Series

R is a ring

$R((x))=\{a_r x^r + a_{r+1}x^{r+1}+a_{r+2}x^{r+2}+\cdots \mid r\in\mathbb{Z},\ a_n\in R\ \forall n \geq r\}$

Example

$$\sum_{n=-5}^{\infty} nx^n = -5x^{-5}-4x^{-4}-3x^{-3}+\cdots \in \mathbb{Z}((x))$$

But $\sum_{n\in\mathbb{Z}} x^n$ is **not**

$A=\cdots+x^{-3}+x^{-2}+x^{-1}+1+x+x^2+x^3+\cdots$

This thing is weird. You might try

$$A=(1+x+x^2+x^4+\cdots)+(x^{-1}+x^{-2}+x^{-3}+\cdots)=\frac{1}{1-x}+\frac{x^{-1}}{1-x^{-1}}=0???$$

Notice that if $f(x)\in F((x))$ we can write $f(x)=\frac{A(x)}{x^k}$ where $k\in\mathbb{N}$ and $A(x)\in R[[x]]$

Addition and multiplication defined as with FPS. Distributive law / collect like terms

OR

$$f(x)=\frac{A(x)}{x^k},\qquad g(x)=\frac{B(x)}{x^l},\qquad A(x),B(x)\in R[[x]]$$

$$f(x)+g(x)=\frac{x^lA(x)+x^kB(x)}{x^{k+l}},\qquad f(x)g(x)=\frac{A(x)B(x)}{x^{k+l}}$$

- If R is an integral domain then $R((x))$ is an integral domain
- If R is a field, then $R((x))$ is a field.

  To prove the second of these, define the **index** of $f(x)\in R((x))$ to be the smallest number $n\in\mathbb{Z}$ such that $[x^n]f(x)\neq 0$.

Example

If $f(x)=\sum_{n=-5}^{\infty} nx^n$, then $I(f)=-5$

Note that if $f\neq 0$ then $x^{-I(f)}f(x)\in R[[x]]$

If R is a field, then $[x^0]x^{-I(f)}f(x)=[x^{I(f)}]f(x)\neq 0$

$\therefore x^{-I(f)}f(x)$ is invertible in $R[[x]]$

$\left(x^{-I(f)}f(x)\right)^{-1}x^{-I(f)}$ is the inverse of $f(x)$

This proves that any non-zero element of $R((x))$ is invertible if R is a field.

Relationships Between These Constructions

$R\subseteq R[x]\subseteq R[[x]]\subseteq R((x))$

If R is an integral domain, $R[x]\subseteq R(x)$

If R is a field, $R(x)\subseteq R((x))$

Every element of $R(x)$ is of the form $f(\frac{x}{g(x)}$, $f(x),g(x)\in R[x]\Rightarrow f(x),g(x)\in R((x))$

$\therefore \frac{f(x)}{g(x)}\in R((x))$ because $R((x))$ is a field

Composition

$A(x),B(x)$ is $A(B(x))$ defined?

If $A(x)\in R[x],\ A(B(x))$ is defined for $B(x)\in R[x]$ or $R(x)$ or $R[[x]]$ or $R((x))$

Why? $A(x)$ is a polynomial $\Rightarrow$ involves finitely many operations $(+,\cdot,-)$. Partition these on $B(x)$

e.g. $A(x)=x^2, A(B(x))=B(x)B(x)$ is defined.

If $A(x)\in R(x),\ A(x)=\frac{f(x)}{g(x)},\ A(B(x))=\frac{f(B(x))}{g(B(x))}$ may or may not be defined

If $A(x)\in R[[x]],\ B(x)\in R[[x]]$

Write $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $\quad B(x) = \sum_{(n=0)}^{\infty} b_n x^n$

$A(B(x)) = \sum_{n=0}^{\infty} a_n (B(x))^n = \sum_{n=0}^{\infty} a_n (b_0 + b_1 x + b_2 x^2 + \cdots)^n = \sum_{n=0}^{\infty} a_n (b_0^n + \cdots)$

If $b_0 \neq 0$, $[x^0]A(B(x))$ is $\sum_{(n=0)}^{\infty} a_n b_0^n$ infinite sum

Declare this to be undefined

What if $b_0 = 0$?

$A(B(x)) = \sum_{(n=0)}^{\infty} a_n (b_1 x + b_2 x^2 + b_3 x^3 + \cdots)^n = \sum_{n=0}^{\infty} a_n \left(\sum_{j=1}^{\infty} b_j x^j\right)^n$

$= \sum_{n=0}^{\infty} a_n \left(\sum_{j_1=1}^{\infty} b_{j_1} x^{j_1}\right)\left(\sum_{j_2=1}^{\infty} b_{j_2} x^{j_2}\right) \ldots \left(\sum_{j_n=1}^{\infty} b_{j_n} x^{j_n}\right)$

Restart

$A(x) = \sum_{n=0}^{\infty} a_n x^n$, $\quad B(x) = \sum_{n=0}^{\infty} b_n x^n$

Assume $b_0 = 0$, $B(x) = \sum_{j=1}^{\infty} b_j x^j$

$A(x) = a_0 + \sum_{k=1}^{\infty} a_k x^k$

$A(B(x)) = a_0 + \sum_{k=1}^{\infty} a_k B(x)^k = a_0 + \sum_{k=1}^{\infty} a_k \left(\sum_{j_1=1}^{\infty} b_{j_1} x^{j_1}\right)\left(\sum_{j_2=1}^{\infty} b_{j_2} x^{j_2}\right) \cdots \left(\sum_{j_k=1}^{\infty} b_{j_k} x^{j_k}\right)$

$= a_0 + \sum_{k=1}^{\infty} \sum_{j_1, j_2, \ldots, j_k \geq 1} a_k b_{j_1} b_{j_2} \cdots b_{j_k} x^{j_1 + j_2 + \cdots + j_k} = a_0 + \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \sum_{\substack{j_1, \ldots, j_k \geq 1 \\ j_1 + \cdots + j_k = n}} a_k b_{j_1} b_{j_2} \ldots b_{j_k} \right) x^n$

The coefficient on each x^n is a finite sum. This nasty expression therefore defines $A(B(x))$

Exercise

If R is a field, $A(x) \in R((x))$, $B(x) \in R[[x]]$, $[x^0]B(x) = 0$, $B(x) \neq 0$

Then $A(B(x))$ is defined in $R((x))$.

Hint: Write $A(x) = -x^{I(A)}\left(x^{-I(A)}A(x)\right)$ use composition for $R(x)$ and $R[[x]]$

Properties

$A_1(x) + A_2(x) = A_3(x) \Rightarrow A_1(B(x) + A_2(B(x)) = A_3(B(x))$

$A_1(x)A_2(x) = A_3(x) \Rightarrow A_1(B(x))A_2(B(x)) = A_3(B(x))$

Special Series & LIFT

October-28-13 10:44 AM

Exponential Series

Define

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \in \mathbb{Q}[[x]]$$

Also written e^x

Logarithm Series

Define

$$\log\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{1}{k} x^k \in \mathbb{Q}[[x]]$$

Binomial Series

Recall that

$$\binom{y}{n} = \frac{y(y-1)\dots(y-n+1)}{n!} \in \mathbb{Q}[y]$$

$$(1+x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n \in \mathbb{Q}[y][[x]]$$

$\mathbb{Q}[y][[x]]$ is the ring of formal power series in x with coefficients in $\mathbb{Q}[y]$

Since polynomials can be evaluated, this implies

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \in \mathbb{C}[[x]]$$

for any $\alpha \in \mathbb{C}$

Lagrange Implicit Function Theorem (LIFT)

Simplified Version

★ Memorize This

Let $\mathbb{K}$ be a commutative ring that contains $\mathbb{Q}$. Let $G(u) \in \mathbb{K}[[u]]$ be a formal power series.

a) There is a unique FPS $W(x) \in \mathbb{K}[[x]]$
   $W(x) = xG(W(x))$
b) $[x^0]W(x) = 0$ and
   $[x^n]W(x) = \frac{1}{n}[u^{n-1}]G(u)^n, \quad \text{for } n \geq 1$

However, we are only going to prove it in the case where $G(u)$ is invertible.

If $G(u)$ is invertible, $W(x) \neq 0$

Remark

If we have a FPS equation in multiple variables, we may be able to use LIFT by treating one variable as the "active" variable, and the other as constants. This may or may not be a choice.

What does this mean?

- These FPS are inspired by Taylor series for functions
- Doesn't make sense to compare theories of FPS to Taylor series directly.
- But they satisfy the same identities

Example

$\exp(x+y) = \exp(x)\exp(y) \in \mathbb{Q}[[x]][[y]]$

Note, $\mathbb{Q}[[x]][[y]] = \mathbb{Q}[[y]][[x]] = \mathbb{Q}[[x,y]]$

Proof

$$\text{LHS} = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!}$$

$$\text{RHS} = \left(\sum_{i=0}^{\infty} \frac{x^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{y^j}{j!}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{x^i y^i}{i!\,j!} = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{x^i y^{n-i}}{i!(n-i)!}, \text{letting } n = i+j$$

Example

$(1+x)^y(1+x)^z = (1+x)^{y+z} \in \mathbb{Q}[y,z][[x]]$

Proof, see notes

Example

$$\exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x}$$

$\log(\exp(x)) = x$

Note make sense of this, note that

$$\exp(x) = \frac{1}{1-u}, \qquad u = \left(1 - \frac{1}{\exp(x)}\right), \log\left(\frac{1}{1-\left(1-\frac{1}{\exp(x)}\right)}\right) = x$$

Example LIFT

BRTs

$\mathcal{T}$ = all BRTs

$\omega(T) = (n(T), \tau(T))$

How many BRTs with n nodes?

Let $A = A(x) = \Phi_{\mathcal{T}}^{n}(x)$

We saw that $A = x(1+A)^2$

Solve using LIFT. Have $G(u) = (1+u)^2$

$$\therefore [x^n]A(x) = \frac{1}{n}[x^{n-1}]G(u)^n = \frac{1}{n}(u^{n-1})(1+u)^{2n} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

What is the average number of terminals among all BRTs with n nodes?

To do this, we had weight function $\omega: T \to \mathbb{N}^2$, $\omega(T) = (n(T), \tau(T))$

Let $A = A(x,y) = \Phi_{\mathcal{T}}^{\omega}(x,y)$

Saw before $A - xy = x(1+A)^2 - x \quad (*)$

Need to compute $[x^n]\frac{\partial}{\partial y}A(x,y)\Big|_{y=1}$

Rewrite $(*)$ as $A = x((1+A)^2 - 1 + y)$

This is the right form to use LIFT. Here, $\mathbb{K} = \mathbb{Q}[y]$

$G(u) = (1+u)^2 - 1 + y = y + 2u + u^2 \in \mathbb{K}[[u]] = \mathbb{Q}[y][[u]]$

$$\therefore [x^n]A(x,y) = \frac{1}{n}[u^{n-1}]G(u)^n = \frac{1}{n}[u^{n-1}]\left((1+u)^2 - 1 + y\right)^n$$

$$[x^n]\frac{\partial}{\partial y}A(x,y) = \frac{1}{n}[u^{n-1}]\frac{\partial}{\partial y}\left((1+u)^2 - 1 + y\right)^n$$

$$= \frac{1}{n}[u^{n-1}]n\left((1+u)^2 - 1 + y\right)^{n-1} = [u^{n-1}]\left((1+u)^2 - 1 + y\right)^{n-1}$$

$$[x^n]\frac{\partial}{\partial y}A(x,y)\Big|_{y=1} = [u^{n-1}]\left((1+u)^2 - 1 + 1\right)^{n-1} = [u^{n-1}](1+u)^{2n-2}$$

$$= \binom{2n-2}{n-1}$$

The average number of terminals is

$$\frac{\binom{2n-2}{n-1}}{\frac{1}{n+1}\binom{2n}{n}}$$

Formal Derivative & Integrals

October-30-13 10:56 AM

Formal Derivatives and Integrals

If $f(x) \in \sum_{n=I(f)}^{\infty} a_n x^n \in R((x))$

1. Formal derivative:
   $$f'(x) = \frac{d}{dx} f(x) = \sum_{n=I(f)}^{\infty} n a_n x^{n-1}$$
2. Formal Integral. Assume $\mathbb{Q} \subseteq R$ and $[x^{-1}]f(x) = 0$
   $$\int f(x)dx = \sum_{\substack{n \geq I(f) \\ n \neq -1}} \frac{a_n}{n+1} x^{n+1}$$
   $n \neq -1$ since $\int x^{-1} dx$ is not a FLS
3. Formal Residue
   $[x^{-1}]f(x)$

Everything you would expect to be true about formal derivatives/integrals is true.

- Formal derivative is R-linear
- Product Rule
- Quotient rule (when defined)
- Chain Rule
- Fundamental Theorem of Calculus
  $$\int f'(x)dx = f(x) + C, \qquad (\text{in fact, } C = -[x^0]f(x))$$
- Integration by parts
  $$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \ + C$$
  True when LHS is defined ($\Rightarrow$ RHS is defined)
- Change of Variables (Substitution Rule)
  $$\int f(g(x))g'(x)dx = F(g(x)) + C$$
  where $F(x) = \int f(x)dx$
  when defined.

Properties of Formal Residue

The formal residue also behaves like a definite integral.

- Fundamental Theorem of Calculus
  $[x^{-1}]f'(x) = 0$
- Integration By Parts
  $[x^{-1}]f(x)g'(x) = -[x^{-1}]f'(x)g(y)$
- Change of Variables
  If $f(x) \in R((x))$, $g \in R[[x]]$, $[x^0]g(x) = 0$, $[x^{I(g)}]g(x)$ must be invertible.
  Then $[x^{-1}]f(g(x))g'(x) = I(g)[x^{-1}]f(x)$

Proof of Properties of Formal Residue

Let $f(x) = \sum_{n \geq I(f)} a_n x^n$, $\quad f'(x) = \sum_{n \geq I(f)} n a_n x^{n-1}$

- $[x^{-1}]f'(x) = 0a_0 = 0$
- $[x^{-1}]\frac{d}{dx}(f(x)g(x)) = 0$
  $[x^{-1}](f'(x)g(x) + f(x)g'(x)) = 0$
- Let $a_n = [y^n]f(y)$
  $h(y) = f(y) - a_{-1}y^{-1}$
  $\therefore [y^{-1}]h(y) = 0$
  Let $H(y) = \int h(y)dy$, $\quad H'(y) = h(y)$
  Let $k(x) = x^{-I(g)}g(x) \in R[[x]]$

  RHS $= I(g)[y^{-1}]f(y) = I(g)a_{-1}$
  LHS $= [x^{-1}]f(g(x))g'(x) = [x^{-1}]\left(h(g(x)) + a_{-1}g(x)^{-1}\right)g'(x)$
  $$= [x^{-1}]\left(H'(g(x))g'(x) + a_{-1}g(x)^{-1}g'(x)\right)$$
  $$= [x^{-1}]\frac{d}{dx}H(g(x)) + [x^{-1}]a_{-1}\frac{g'(x)}{g(x)}$$
  $$= 0 + [x^{-1}]a_{-1}\frac{\frac{d}{dx}\left(x^{I(g)}k(x)\right)}{x^{I(g)}k(x)}$$
  $$= [x^{-1}]a_{-1}\frac{I(g)x^{I(g)-1}k(x) + x^{I(g)}k'(x)}{x^{I(g)}k(x)}$$
  $$= [x^{-1}]a_{-1}\left(I(g)x^{-1} + \frac{k'(x)}{k(x)}\right) = a_{-1}I(g)$$
  Since $\frac{k'(x)}{k(x)} \to FPS \to 0$

LIFT Proof & Comp. Inv.

10:48 AM

Lemma

Let $A(x), B(x) \in R[[x]]$ with $[x^0]A(x) = [x^0]B(x) = 0$
Assume $[x]B(x)$ is invertible in R. If $A(B(x)) = 0$ then $A = 0$

Theorem

$f(x) \in R[[x]], \qquad g(u) \in R[[u]]$
Suppose $[x^0]f(x) = 0 \Rightarrow g(f(x))$ is defined.
If $g(f(x)) = x$ then the following are true

i) $[u^0]g(u) = 0$
   $[u]g(u) \cdot [x]f(x) = 1$
ii) $f(g(u)) = u$
iii) By i), $G(u) = \frac{u}{g(u)} \in R[[u]]$ is defined.
   Then $f(x)$ is a solution to the LIFT equation: $f = xG(f)$

Compositional Inverse

If $g(f(x)) = x$ we say that f is the **compositional inverse** of g.

Notes

i) gives necessary conditions for the compositional inverse of g to exist if the constant term is 0.
ii) $\Rightarrow$ is also the composition inverse of f
iii) can use LIFT to solve for the compositional inverse of g.

Corollary

If $G(u)$ is invertible then the LIFT equation $W = xG(W)$ can be rewritten as $W\left(\frac{u}{G(u)}\right) = u$

General Statement of LIFT

- $\mathbb{K}$ is a ring containing $\mathbb{Q}$
- $F(u), G(u) \in \mathbb{K}[[u]]$

i) There is a unique FPS $W(x) = \mathbb{K}[[x]]$ such that
   $W(x) = xG(W(x))$

ii) $[x^0]F(W(x)) = F(0)$
   $$[x^n]F(W(x)) = \frac{1}{n}[u^{n-1}]F'(u)G(u)^n, \qquad n \geq 1$$

Notes

- The special case given before is $F(u) = u$. This is the common way to use LIFT
- In our proof, we'll assume $G(u)$ is invertible.

Proof of Lemma (Compositional Inverse)

Exercise. Check that $[x^{I(A)}]A(B(x)) \neq 0$ if $A \neq 0$

Proof of Theorem

i) Note that for any FPS $A(x)$,
   $[x^0]A(x) = A(0)$
   $[x]A(x) = A'(0)$
   $[u^0]g(u) = g(0) = g(f(0)) = 0$ using $f(0) = 0$ and $g(f(x)) = x$
   $g(f(x)) = x$
   $g'(f(x))f'(x) = 1$
   $g'(0)f'(0) = 1$
ii) Let $A(u) = f(g(u)) - u$
   $A(f(x)) = f\left(g(f(x))\right) - f(x) = f(x) - f(x) = 0$
   Since $[x]f(x)$ is invertible (by 1), by the lemma, $A = 0$

   Note: Here we used composition is associative. Have not proved this but it is true
iii) Check that $f = xG(f)$
   $$\text{RHS} = xG(f(x)) = x\frac{f(x)}{g(f(x))} = x\frac{f(x)}{x} = f(x) = \text{LHS}$$

■

Proof of Corollary

Both equations say that $W(x)$ is the compositional inverse of $\frac{u}{G(u)}$

Proving LIFT

Strategy for proving LIFT: use Formal residues and change of variables $x = \frac{u}{G(u)}$

Proof of General Statement of LIFT

i) Write $G(u) = \sum_{n=0}^{\infty} a_n u^n, \qquad W(x) = \sum_{n=0}^{\infty} b_n x^n$
   $W(x) = xG(W(x))$
   $W(0) = 0, \qquad G(W(0)) = 0$
   $\therefore b_0 = 0$
   Now compute coefficients
   $[x^n]W(x) = [x^n]xG(W(x)) = [x^{n-1}]G(W(x))$
   $$b_n = \sum_{k=1}^{n-1} \sum_{\substack{j_1,\ldots,j_k \geq 1 \\ j_1+\cdots+j_k = n-1}} a_k b_{j_1} \ldots b\_j_k$$
   RHS involves $b_1, \ldots, b_{n-1}$
   This lets us solve for $b_1, b_2, b_3, \ldots$ recursively
   $\Rightarrow W(x)$ exists and is unique.

ii) Turn into a formal residue:
   $\Rightarrow [x^n]F(W(x)) = [x^{-1}]x^{-n-1}F(W(x))$
   Let $f(x) = x^{-n-1}F(W(x))$
   $\Rightarrow [x^n]F(W(x)) = [x^{-1}]f(x)$

   Let $g(u) = \frac{u}{G(u)}$
   Note
   - $I(g) = 1$
   - ? $W(g(u)) = u$ (Proved above)

   Change of variables:
   $$\Rightarrow [x^{-1}]f(x) = \frac{1}{I(g)}[u^{-1}]f(g(u))g'(u)$$
   $$= [u^{-1}]g(u)^{-n-1}F\left(W(g(u))\right)g'(u) = [u^{-1}]g(u)^{-n-1}F(u)g'(u)$$
   $$= [u^{-1}]g(u)^{-n-1}g'(u)F(u) = [u^{-1}]\frac{d}{du}\left(-\frac{1}{n}g(u)^{-n}\right)F(u)$$
   $$= -\frac{1}{n}[u^{-1}]\frac{d}{du}(g(u)^{-n})F(u) = \frac{1}{n}[u^{-1}]g(u)^{-n}F'(u)$$
   ? Last step is integration by parts
   $$\frac{1}{n}[u^{-1}]g(u)^{-n}F'(u) = \frac{1}{n}[u^{-1}]\left(\frac{u}{G(u)}\right)^{-n}F'(u)$$
   $$= \frac{1}{n}[u^{-1}]u^{-n}G(u)^nF'(u) = \frac{1}{n}[u^{n-1}]G(u)^nF'(u)$$
   ■

LIFT Examples

November-04-13 10:48 AM

Example

Let $\mathcal{P}$ be the set of all PPTs. For $T \in \mathcal{P}$, let $\omega(T) = \big(n(T), d(T)\big)$ where
$d(T) = \deg(\odot) =$ degree of the root
$n(T) = \#$ of nodes
Compute $\Phi_{\mathcal{P}}^{\omega}(x, y)$

Solution

Use the bijection $\mathcal{P} \rightleftarrows \{\odot\} \times \mathcal{P}^*,\ T \leftrightarrow (0, c_1, \dots, c_k)$
This is not weight-preserving for ω.
We have $n(T) = n(\odot) + n(c_1) + \cdots + n(c_k),\ d(T) = k$
Define the weight functions:

$\nu: \mathcal{P} \to \mathbb{N}^2, \quad \nu(c) = (n(c), 1)$
$\mu: \{\odot\} \to \mathbb{N}^2, \quad \mu(\odot) = (1, 0)$

$\therefore \omega(T) = \mu(\odot) + \nu(c_1) + \cdots + \nu(c_k)$

$$\therefore \Phi_{\mathcal{P}}^{\omega}(x, y) = \Phi_{\{\odot\}}^{\mu}(x, y) \cdot \frac{1}{1 - \Phi_{\mathcal{P}}^{\nu}(x, y)} = x \cdot \frac{1}{1 - \Phi_{\mathcal{P}}^{\nu(x,y)}}$$

To compute $\Phi_{\mathcal{P}}^{\omega}(x, y)$ we need to know $\Phi_{\mathcal{P}}^{\nu}(x, y)$. But, notice

$$\Phi_{\mathcal{P}}^{\nu}(x, y) = \sum_{t \in \mathcal{P}} x^{n(T)} y^1 = y \sum_{T \in \mathcal{P}} x^{n(T)} = y \Phi_{\mathcal{P}}^{n}(x)$$

We already know $\Phi_{\mathcal{P}}^{n}(x) = A$ where $A = \frac{x}{1-A}$
$\therefore$ We want to compute

$$\Phi_{\mathcal{P}}^{\omega}(x, y) = x \cdot \frac{1}{1 - yA}, \qquad \text{where } A = \frac{x}{1 - A}$$

This is exactly what LIFT lets us do:
Here $\mathbb{K} = \mathbb{Q}[y],\ G(u) = \frac{1}{1-u},\ F(u) = \frac{1}{1-yu}$

$$[x^{n+1}]\Phi_{\mathcal{P}}^{\omega}(x, y) = [x^{n+1}] x \cdot \frac{1}{1 - yA(x, y)} = [x^n] \frac{1}{1 - yA(x, y)} = [x^n] F\big(A(x, y)\big)$$

$$= \frac{1}{n} [u^{n-1}] F'(u) G(u)^n$$

Note: $F'(u) = \frac{\partial}{\partial u} F(u)$

$$[x^{n+1}]\Phi_{\mathcal{P}}^{\omega}(x, y) = \frac{1}{n} [u^{n-1}] \frac{y}{(1 - uy)^2} \cdot \left(\frac{1}{1 - u}\right)^n$$

$$= \frac{1}{n} [u^{n-1}] y \left(\sum_{k=0}^{\infty} (k+1) y^k u^k\right) \left(\sum_{j=0}^{\infty} \binom{n + j - 1}{j} u^j\right) = \frac{1}{n} \sum_{k=0}^{n-1} (k+1) y^{k+1} \binom{2n - k - 2}{n - k - 1}$$

Also, $[x^0]\Phi_{\mathcal{P}}^{\omega}(x, y) = 0$

$$\therefore \Phi_{\mathcal{P}}^{\omega}(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n} (k+1) \binom{2n - k - 2}{n - k - 1} y^{k+1} x^{n+1}$$

Integer Partitions

10:33 AM

Integer Partition

An integer partition is a tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integer such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.
$n(\lambda) := \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is the **size** of λ. (old notation, $|\lambda|$)
$k(\lambda) := k$ is the **length** of λ
The Ferrers diagram, F_λ of λ had λ_i boxes in row i (left-justified)

$d(\lambda) =$# of boxes on main diagonal of $F_\lambda = \#\{\, i : \lambda_i \geq i\}$

Let $\mathcal{Y}$ denote the set of all paritions.
Let $p(n)$ denote the number of partitions of size n.

Theorem

$$p(n) = [x^n] \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

equivalently,

$$\Phi_{\mathcal{Y}}^{n}(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

Note

RHS is an infinite product of FPS. We haven't defined what this means yet.

Note

For a sequence of integers to have a limit, it must be eventually constant.

FPS Limit

Let $A_1(x), A_2(x), A_3(x), \dots \in R[[x]]$

We say that $\lim_{k\to\infty} A_k(x) = \sum_{n=0}^{\infty} a_n x^n$

if there is a function $\phi: \mathbb{N} \to \mathbb{N}$ such that $[x^n]A_k(x) = a_n \ \ \forall k \geq \phi(n)$

Note:

In practice, often $\phi(n) = n$.

Sum and Product Limits

More generally, if $B_1(x), B_2(x), \dots, \in R[[x]]$

define $\sum_{j=1}^{\infty} B_j(x) = \lim_{k=\infty} \sum_{j=1}^{k} B_j(x)$

and $\prod_{j=1}^{\infty} B_j(x) = \lim_{k\to\infty} \prod_{j=1}^{k} B_j(x)$

What does this mean?

$$\prod_{j=1}^{\infty} B_j(x) = \sum_{n=1}^{\infty} b_n x^n$$

means there is a function $\phi: \mathbb{N} \to \mathbb{N}$ such that

$$b_n = [x^n] \prod_{j=1}^{k} B_j(k) \ \ \forall k \geq \phi(n)$$

Multiplicity Sequence

For a partition $\lambda \in \mathcal{Y}$, let $m(\lambda) = \langle m_1(\lambda), m_2(\lambda), m_3(\lambda), \dots \rangle$
where $m_j(\lambda) = |\{i : \lambda_i = j\}|$

Proposition

Let $\mathcal{M}$ be the set of all nonnegative integer sequences: $\rho = \langle r_1, r_2, r_3, \dots \rangle$ such that $r_i \neq 0$ only for finitely many i.

The functions $\lambda \mapsto m(\lambda)$ is a bijection $\mathcal{Y} \rightleftarrows \mathcal{M}$
Furthermore, if $\lambda(\lambda_1, \dots, \lambda_k) \in \mathcal{Y}$ corresponds to $m(\lambda) = e = \langle r_1, \dots, r_l \rangle$ then

a) $k(\lambda) = r_1 + r_2 + \cdots + r_l$
b) $n(\lambda) = r_1 + 2r_2 + 3r_3 + \cdots + lr_l$
c) $\lambda_1 = l(p)$
d) $d(\lambda) = \max\{j : j \leq r_1 + r_{j+1} + \cdots + r_l\}$

Theorem

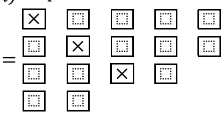
$$\Phi_{\mathcal{Y}}^{(n,k)}(x, y) = \sum_{\lambda\in\mathcal{Y}} x^{n(\lambda)} y^{k(\lambda)} = \prod_{j=1}^{\infty} \frac{1}{1 - x^j y} \text{ in } \mathbb{Q}[[y]][[x]]$$

not in $\mathbb{Q}[[x]][[y]]$

Example Integer Partitions

Example 1

$\lambda = 5\,5\,4\,2$
$n(\lambda) = 5 + 5 + 4 + 2 = 10$
$k(\lambda) = 4$

$F_\lambda =$ 

$d(\lambda) = 3$

Example 2

$p(4) = 5$
1 1 1 1
2 1 1
3 1
2 2
4

Example FPS Limits

$\lim_{n\to\infty} x^n = 0$
Why? Because if $k \geq n + 1$, $[x^n]x^k = 0$

$$A_k(x) = \sum_{n=0}^{k} a_n x^n$$

Then $\lim_{k\to\infty} A_k(x) = \sum_{n=0}^{\infty} a_n x^n$
Why? If $k \geq n$ then $[x^n]A_k(x) = a_n$
Note: This is how we defined infinite sums in calculus.

Example

$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \cdots = \prod_{j=1}^{\infty} \left(1 + x^{2^{j-1}}\right)$$

Claim

This is equal to $\frac{1}{1-x} = \sum_{n=0}^{\infty} 1x^n$
To prove this, note that for $k \geq n$,

$$[x^n] \prod_{i=1}^{k} \left(1 + x^{2^{j-1}}\right) = [x^n]\left(1 + x + x^2 + \cdots + x^{2^{k-1}}\right) = 1$$

Proof of Theorem (Idea)

Main Idea

Think of 5542 as $\langle 0, 1, 0, 1, 2 \rangle$ meaning 0 parts of size 1, 1 part of size 2, etc.

Let $\mathcal{M}$ be the set of all nonnegative integer sequences: $\rho = \langle r_1, r_2, r_3, \dots \rangle$ such that $r_i \neq 0$ only for finitely many i.

Example

$\langle 3, 1, 5, 0, 1, 0, 0, 0, 0, 0, 0, \dots \rangle \in \mathcal{M}$
Convention: Drop infinitely many 0s at the end of the notation.
Write $\langle r_1, r_2, \dots, r_l \rangle$ if $r_j = 0 \forall j > l$
$\Rightarrow \rho = \langle 3, 1, 5, 0, 1 \rangle$

Let $l(\rho) = \max\{j : r_j \neq 0\}$, $l(\langle 0, 0, 0, \dots \rangle) = 0$

Proof of Proposition

Exercise

Proof of Theorem (Actual Proof)

We must show that

$$\Phi_{\mathcal{Y}}^{n}(x) = \sum_{j=1}^{\infty} \frac{1}{1 - x^j}$$

$$\Phi_{\mathcal{Y}}^{n}(x) = \sum_{\lambda\in\mathcal{Y}} x^{n(\lambda)} = \sum_{\rho\in\mathcal{M}} x^{r_1 + 2r_2 + 3r_3 + \cdots} = \left(\sum_{r_1\in\mathbb{N}} x^{r_1}\right)\left(\sum_{r_2\in\mathbb{N}} x^{2r_2}\right)\left(\sum_{r_3\in\mathbb{N}} x^{3r_3}\right)$$

$$= \left(\frac{1}{1 - x}\right)\left(\frac{1}{1 - x^2}\right)\left(\frac{1}{1 - x^3}\right) \cdots = \prod_{j=1}^{\infty} \left(\frac{1}{1 - x^j}\right)$$

Need to justify

$$\sum_{\rho\in M} x^{r_1 + 2r_2 + 3r_r + \cdots} = \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

Let $k \geq n$. We'll show that

$$[x^n] \sum_{\rho\in M} x^{r_1 + 2r_2 + 3r_3 + \cdots} = [x^n] \prod_{j=1}^{k} \frac{1}{1 - x^j}$$

$$\text{RHS} = [x^n]\prod_{j=1}^{k}\left(\sum_{r_j=0}^{\infty}(x^j)^{r_i}\right) = [x^n]\sum_{(r_1,r_2,\dots,r_k)\in\mathbb{N}^k} x^{r_1+2r_2+\cdots+kr_k}$$
$$= \left|\left\{(r_1,\dots,r_k)\in\mathbb{N}^k : r_1+2r_2+3r_3+\cdots kr_k = n\right\}\right|$$
$\text{LHS} = |\{\rho = \langle r_1, r_2, \dots\rangle \in \mathcal{M} : r_1 + 2r_2 + 3r_3 + \cdots = n\}$

If $\rho = m(\lambda)$ and $r_1 + 2r_2 + 3r_3 + \cdots = n$ then $n(\lambda) = n$

$k \geq n = n(\lambda) \geq l(\rho) \Rightarrow r_{k+1} = r_{k+2} = \cdots = 0$

$\therefore LHS = \left|\left\{(r_1,\dots,r_k)\in\mathbb{N}^k : r_1 + 2r_2 + \cdots + kr_k = n\right\}\right|$

Once we know the general method we can do all sorts of variations.

Proof of Theorem

$$\Phi_{\mathcal{Y}}^{(n,k)}(x,y) = \sum_{\lambda\in\mathcal{Y}} x^{n(\lambda)}y^{k(\lambda)} = \sum_{\rho\in\mathcal{M}} x^{r_1+2r_2+3r_3+\cdots}y^{r_1+r_2+r_3+\cdots}$$
$$= \sum_{\rho\in\mathcal{M}}(xy)^{r_1}(x^2y)^{r_2}(x^3y)^{r_3}\dots = \sum_{r_1\in\mathbb{N}}(xy)^{r_1}\sum_{r_2\in\mathbb{N}}(x^2y)^{r_2}\sum_{r_3\in\mathbb{N}}(x^3y)^{r_3}\dots$$
$$= \prod_{j=1}^{\infty}\frac{1}{1-x^jy}$$
■

Integer Partitions Cont.

November-11-13 10:51 AM

Partition with Distinct Parts

A partition λ is said to have distinct parts if $m_j(\lambda) \in \{0,1\}$ for all $j \in 1,2,3,\ldots$
Also called "strict" partitions.

Let $\mathcal{D} \subseteq \mathcal{Y}$ be the set of all partitions with distinct parts.

Theorem

$$\Phi_{\mathcal{D}}^{(n,k)}(x,y) = \prod_{j=1}^{\infty}(1 + x^j y)$$

Notation

Not in course notes

If $\phi(\lambda)$ is a statement about λ, let $\mathcal{Y}_{\phi(\lambda)}$ be the set of all partitions $\lambda \in \mathcal{Y}$ for which $\phi(\lambda)$ is true.
Do same thing with $\mathcal{D}$

Conjugate of a Partition

For $\lambda \in \mathcal{Y}$ the conjugate of λ, denoted $\tilde{\lambda}$ is the partition whose Ferrers diagram is the transpose of F_λ

Example

$\lambda = 6\,6\,3\,1$, $\quad \tilde{\lambda} = 4\,3\,3\,2\,2\,2$

Note

$n(\lambda) = n(\tilde{\lambda})$
$k(\lambda) = \tilde{\lambda}_1$
$\lambda_1 = k(\tilde{\lambda})$
$d(\lambda) = d(\tilde{\lambda})$

Shifted Diagram

The sifted diagram of $\lambda \in \mathcal{D}$ is the Ferrers diagram, with i^{th} row shifted $i-1$ rows right.

Example Partitions with Distinct Parts

$7\,5\,4\,2 \in \mathcal{D}$, $\quad 7\,5\,5\,4\,2 \notin \mathcal{D}$

Proof of Theorem

$$\Phi_{\mathcal{D}}^{(n,k)}(x,y) = \sum_{\lambda \in \mathcal{D}} x^{n(\lambda)} y^{k(\lambda)} = \sum_{\substack{\rho \in \mathcal{M} \\ r_i \in \{0,1\} \\ \forall i}} x^{r_1 + 2r_2 + 3r_3 + \cdots} y^{r_1 + r_2 + r_3 + \cdots}$$

$$= \sum_{r_1 \in \{0,1\}} (xy)^{r_1} \sum_{r_2 \in \{0,1\}} (x^2 y)^{r_2} \sum_{r_3 \in \{0,1\}} (x^3 y)^{r_3} \ldots = (1+xy)(1+x^2y)(1+x^3y)\ldots$$

■

Example of Notation

$\mathcal{Y}_{\lambda_1 \le l}$ is the set of all partitions λ such that $\lambda_1 \le l$.

- $\mathcal{Y}_{\lambda_1 \le l} : \Phi_{\mathcal{Y}_{\lambda_1 \le l}}^{(n,k)} = \prod_{j=1}^{l} \frac{1}{1 - x^j y}$
- $\mathcal{Y}_{\lambda_1 = l} : \Phi_{\mathcal{Y}_{\lambda_1 = 1}}^{(n,k)}(x,y) = \left(\frac{1}{1-xy}\right)\left(\frac{1}{1-x^2y}\right)\cdots\left(\frac{1}{1-x^{l-1}y}\right)\left(\frac{1}{1-x^l y}\right)$
  $= x^l y \prod_{j=1}^{l} \frac{1}{1-x^j y}$
- $\mathcal{D}_{\lambda_1 \le l} : \Phi_{\mathcal{D}_{\lambda_1 \le l}}^{(n,k)}(x,y) = \prod_{j=1}^{l}(1 + x^j y)$
- $\mathcal{D}_{\lambda_1 = l} : \Phi_{\mathcal{D}_{\lambda_1 = l}}^{(n,k)}(x,y) = x^l y \prod_{j=1}^{l-1}(1 + x^j y)$

☐ Exercise: Check these

Some Tricks

Example

How may partitions of n with k parts.

Method 1

$$[x^n y^k] = \Phi_{\mathcal{Y}}^{(n,k)}(x,y) = [x^n y^k] \prod_{j=1}^{\infty} \frac{1}{1 - x^j y}$$

Method 2

Consider $\mathcal{Y}_{k(\lambda)=k} \leftarrow$ exactly k parts.
Want to compute $[x^n]\Phi_{\mathcal{Y}_{k(\lambda)=k}}^{n}(x)$

$$\Phi_{\mathcal{Y}_{k(\lambda)=k}}^{n}(x) = \Phi_{\mathcal{Y}_{\lambda_1 = k}}^{n}(x) = x^k \prod_{j=1}^{k}\left(\frac{1}{1-x^j}\right)$$

Combining methods 1 and 2:

$$\boxed{\prod_{j=1}^{\infty} \frac{1}{1 - x^j y} = \sum_{k=0}^{\infty}\left(x^k \prod_{j=1}^{k} \frac{1}{1-x^j}\right) y^k}$$

Example

How many partitions of n with k distinct parts.

Method 1

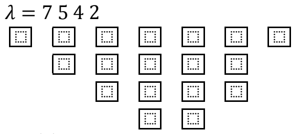
$$[x^n y^k]\Phi_{\mathcal{D}}^{(n,k)}(x,y) = \prod_{j=1}^{\infty}(1 + x^j y)$$

Method 2

Consider $\mathcal{D}_{k(\lambda)=k}$
Want $[x^n]\Phi_{\mathcal{D}_{k(\lambda)=k}}^{n}(x)$

Example Shifted Diagram

$\lambda = 7\,5\,4\,2$



If $k(\lambda) = n$
First k columns have $\frac{k(k+1)}{2}$ boxes, call this shape Δ_k
The remaining columns form the Ferrers diagram of a partition with $\le k$ parts.
The conjugate of this has $\tilde{\lambda}_1 \le k$.

Get a bijection
$\mathcal{D}_{k(\lambda)=k} \rightleftarrows \{\Delta_k\} \times \mathcal{Y}_{\lambda_1 \le k}$

$$\boxed{\Phi_{\mathcal{D}_{k(\lambda)=k}}(x) = x^{\frac{k(k+1)}{2}} \prod_{j=1}^{k}\left(\frac{1}{1-x^j}\right)}$$

Exponential Generating Functions

November-15-13 10:43 AM

Class of Structures

A class of structures (or species) $\mathcal{A}$ is a rule that assigns to each finite set X another finite set $\mathcal{A}_X$, called the set of $\mathcal{A}$-structures on X.
There are conditions:
1) If $X \rightleftarrows Y$ then $\mathcal{A}_X \rightleftarrows \mathcal{A}_Y$
2) Technical, we won't need them in this course.

In general there will be some pictorial way to visualize an $\mathcal{A}$-structure on X where the elements of X are labels in the picture.
The technical conditions (2) basically say that the bijection $\mathcal{A}_X \rightleftarrows \mathcal{A}_Y$ comes from relabelling.

Note that if $|X| = n$ then $\mathcal{A}_X \rightleftarrows \mathcal{A}_{N_n}$

Notation

Since it is cumbersome to write $\mathcal{A}_{N_n}$ we usually write
$\mathcal{A}_n \coloneqq \mathcal{A}_{N_n}$
If we know $|\mathcal{A}_n|$ for all n, we know $|\mathcal{A}_X|$ for any finite set X.

Exponential Generating Function

The exponential generating function for $\mathcal{A}$ is the FPS

$$A(x) = \sum_{n=0}^{\infty} |\mathcal{A}_n| \frac{x^n}{n!}$$

Hence, if $|X| = n$, then $|\mathcal{A}_X| = n!\,[x^n]A(x)$

Cyclic Permutation

A permutation is cyclic if its diagram has one component.

$\mathcal{C}_X$ is the set of all cyclic permutations of X
(Note $\mathcal{C}_X \subseteq S_X$ for all finite sets X)

Example: Trees

As mentioned on day 1, there is no set of trees.
However, we have the species of trees $\mathcal{T}$ assigns to each finite set X the set $\mathcal{T}_X$ of all trees with vertex set X.

Note

Not rooted

Example

$\mathcal{T}_{\{1,2,3\}} = \{1-2-3, 1-3-2, 3-2-1\}$
$\mathcal{T}_{\{4,5,6\}} = \{5-6-7, 5-7-6, 6-5-7\}$
Note $\{1,2,3\} \rightleftarrows \{4,5,6\}$ and $\mathcal{T}_{\{1,2,3\}} \rightleftarrows \mathcal{T}_{\{5,6,7\}}$

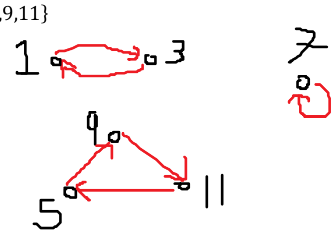
Some Basic Examples

Permutations. (S)
For a finite set X, S_X is the set of all bijections $\sigma: X \to X$ (permutations of X)
$S_n = S_{N_n} = \{\sigma: N_n \to N_n : \sigma \text{ a bijection }\}$
$|S_n| = n!\ \forall n \in \mathbb{N}$
$\therefore$ the exponential generating function is

$$S(x) = \sum_{n=0}^{\infty} |S_n| \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} n! \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

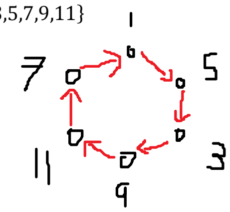
Pictorial Representation

E.g. $X = \{1,3,5,7,9,11\}$
$\sigma(1) = 3$
$\sigma(3) = 1$
$\sigma(5) = 9$
$\sigma(7) = 7$
$\sigma(9) = 11$
$\sigma(11) = 5$



Example Cyclic Permutation

$X = \{1,3,5,7,9,11\}$



$$|\mathcal{C}_X| = \begin{cases} (n-1)! & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$\therefore$ EGF is

$$C(x) = \sum_{n=0}^{\infty} |\mathcal{C}_X| \cdot \frac{x^n}{n!} = \sum_{n=1}^{\infty} (n-1)! \cdot \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$$

Example: Sets $\mathcal{E}$

For any finite set X, $\mathcal{E}_X = \{X\}$
$|\mathcal{E}_X| = 1\ \forall x$
$\therefore$ EGF is

$$E(x) = \sum_{n=0}^{\infty} |\mathcal{E}_n| \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!} = \exp(x)$$

Note

$$E(x) = \exp(X), \quad S(x) = \frac{1}{1-x}, \quad C(x) = \log\left(\frac{1}{1-x}\right)$$

$$E(C(X)) = S(X)$$

Informally:
A permutation is a set of cyclic permutations.

Sum, Product, & Difference of Classes

November-18-13 10:31 AM

Analogs of basic operations on sets: $\cup, \setminus, \times, (\cdot)^*$

Class Equivalence

If $\mathcal{A}$ and $\mathcal{B}$ are classes of structures we'll say $\mathcal{A}$ and $\mathcal{B}$ are equivalent (written $\mathcal{A} \equiv \mathcal{B}$) if, for everry finite set X, $\mathcal{A}_X \rightleftarrows \mathcal{B}_X$.

- Numerical equivalence: (just this) $|\mathcal{A}_X| = |\mathcal{B}_X|$
- Natural equivalence: an additional condition (informally, can use the same pictorial representation)

Sum of Classes

Suppose $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots$ is a finite or infinite list of classes.
First, suppose $\mathcal{A}_X^{(1)}, \mathcal{A}_X^{(2)}, \ldots$ are pairwise disjoint for every finite set X.
(i.e. $\mathcal{A}_X^{(i)} \cap \mathcal{A}_X^{(j)} = \emptyset \; \forall i \neq j$)
Define the sum $\mathcal{B} = \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \cdots$, $\mathcal{B}_X \coloneqq \mathcal{A}_X^{(1)} \cup \mathcal{A}_X^{(2)} \cup \cdots$
i.e. a $\mathcal{B}$-structure on X is an $\mathcal{A}^{(1)}$-structure, or a $\mathcal{A}^{(2)}$-structure, or...
If $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots$ are not pairwise disjoint, make them disjoint by replacing $\mathcal{A}_X^{(i)}$ by $\{i\} \times \mathcal{A}_X^{(i)}$ (Equivalent)

$$\mathcal{B}_X \coloneqq \left(\{1\} \times \mathcal{A}_X^{(1)}\right) \cup \left(\{2\} \times \mathcal{A}_X^{(2)}\right) \cup \cdots$$

Note

We require that $\mathcal{B}_X$ be finite, so only finitely many $\mathcal{A}_X^{(i)}$ may be non-empty for all X.

EGFs

If $\mathcal{B} = \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \cdots$
then $|\mathcal{B}_X| = \left|\mathcal{A}_X^{(1)}\right| + \left|\mathcal{A}_X^{(2)}\right| + \cdots$

EGF for $\mathcal{B}$ is
$B(x) = A^{(1)}(x) + A^{(2)}(x) + \cdots$

Difference of Classes

Suppose $\mathcal{A}, \mathcal{B}$ are classes and $\mathcal{A}_X \subseteq \mathcal{B}_X$ for all X. Then we say $\mathcal{A}$ is a **subclass** of $\mathcal{B}$ and we define the class $\mathcal{B} \setminus \mathcal{A}$
$(\mathcal{B} \setminus \mathcal{A})_X = \mathcal{B}_X \setminus \mathcal{A}_X$

EGF for $\mathcal{B} \setminus \mathcal{A}$ is $B(x) - A(x)$

Product of Classes

Superposition of Classes

$\mathcal{A}, \mathcal{B}$ classes. The superposition is denoted $\mathcal{A}\&\mathcal{B}$ and defined as
$(\mathcal{A}\&\mathcal{B})_X = \mathcal{A}_X \times \mathcal{B}_X$
Note: $|(\mathcal{A}\&\mathcal{B})_X| = |\mathcal{A}_X| \cdot |\mathcal{B}_X|$
but this does not produce a nice EGF formula for $\mathcal{A}\&\mathcal{B}$.

Product of Classes

$\mathcal{A}, \mathcal{B}$ classes. Define $\mathcal{A} * \mathcal{B}$ as follows:

$$(\mathcal{A} * \mathcal{B})_X \coloneqq \bigcup_{S \subseteq X} (\mathcal{A}_S \times \mathcal{B}_{X \setminus S})$$

In other words, an $(\mathcal{A} * \mathcal{B})$-structure on X is a pair (α, β) where α is an $\mathcal{A}$-structure on a subset $S \subseteq X$ and β is a $\mathcal{B}$-structure on its complement $X \setminus S$.

If $|X| = n$,

$$|(\mathcal{A} * \mathcal{B})_X| = \sum_{S \subseteq X} |\mathcal{A}_S||\mathcal{B}_{X \setminus S}| = \sum_{k=0}^{n} \sum_{S \in \mathcal{B}(X,k)} |\mathcal{A}_k||\mathcal{B}_{n-k}| = \sum_{k=0}^{n} \binom{n}{k} |\mathcal{A}_k||\mathcal{B}_{n-k}|$$

$$= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} |\mathcal{A}_k||\mathcal{B}_{n-k}|$$

Rewrite:

$$|(\mathcal{A} * \mathcal{B})_n| \cdot \frac{1}{n!} = \sum_{k=0}^{n} \left(|\mathcal{A}_k| \cdot \frac{1}{k!}\right)\left(|\mathcal{B}_{n-k}| \cdot \frac{1}{(n-k)!}\right)$$

$$\sum_{n=0}^{\infty} |(\mathcal{A} * \mathcal{B})_n| \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(|\mathcal{A}_k| \cdot \frac{1}{k!}\right)\left(|\mathcal{B}_{n-k}| \cdot \frac{1}{(n-k)!}\right) x^n$$

$$= \left(\sum_{k=0}^{\infty} |\mathcal{A}_k| \cdot \frac{x^k}{k!}\right)\left(\sum_{l=0}^{\infty} |\mathcal{B}_l| \cdot \frac{x^l}{l!}\right)$$

$\therefore$ EGF for $\mathcal{A} * \mathcal{B}$ is $A(x)B(x)$

Note

$*$ is essentailly associative:
$(\mathcal{A} * \mathcal{B}) * \mathcal{C} \equiv \mathcal{A} * (\mathcal{B} * \mathcal{C})$

Example: k-sets ($\mathcal{E}_k$)

$$(\mathcal{E}_k)_X = \begin{cases} \{X\} & \text{if } |X| = k \\ \emptyset & \text{otherwise} \end{cases}$$

$$|(\mathcal{E}_k)_X| = \begin{cases} 1 & \text{if } |X| = k \\ 0 & \text{otherwise} \end{cases}$$

EGF

$$E_k(x) = \frac{x^k}{k!}$$

Sum Example

$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3 \oplus \cdots$

Note

If $|X| = k$, then an $\mathcal{E}$-structure on X is an $\mathcal{E}_0$ structure, or an $\mathcal{E}_1$ structure, or an $\mathcal{E}_2$-structure, or ...
but there are no $\mathcal{E}_i$ structures on X for indicies $i \neq k$.
$\therefore$ an $\mathcal{E}$-structure on X is an $\mathcal{E}_k$ structure on X.

$$E(x) = E_0(x) + E_1(x) + E_2(x) + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Example

$\mathcal{C}$ is a subclass of $\mathcal{S}$.
$\mathcal{S} \setminus \mathcal{C}$ is the class of non-cyclic permutations.

Example Product of Classes: $\mathcal{E} * \mathcal{E}$

$$(\mathcal{E} * \mathcal{E})_X = \bigcup_{S \subseteq X} \mathcal{E}_S * \mathcal{E}_{X \setminus S} = \bigcup_{S \subseteq X} \{(S, X \setminus S)\} = \{(S, X \setminus S), \text{where } S \subseteq X\}$$

So $(\mathcal{E} * \mathcal{E})$ is all the ways of partitionning a set into 2 subsets.

What is $\mathcal{E} * \mathcal{E} * \mathcal{E}$?

Technically should have brackets. Either $\mathcal{E} * (\mathcal{E} * \mathcal{E})$ or $(\mathcal{E} * \mathcal{E}) * \mathcal{E}$
Take a set X, split it into 2 parts, then split one of the parts into 2 parts.
Equivalent to splitting into 3 parts.
$(\mathcal{E} * \mathcal{E} * \mathcal{E}) = \{(A, B, C) \text{ where } A \cup B \cup C = X \text{ and } A \cap B = A \cap C = B \cap C = \emptyset\}$

Species of Singletons

Let $\mathcal{X} = \mathcal{E}_1$
If $\mathcal{A}$ is a class, what is $\mathcal{X} * \mathcal{A}$?

$$(\mathcal{X} * \mathcal{A})_X = \bigcup_{S \subseteq X} \mathcal{X}_S \times \mathcal{A}_{X \setminus S}$$

But $\mathcal{X}_S = \emptyset$ if $|S| \neq 1$

$$\therefore (\mathcal{X} * \mathcal{A})_X = \bigcup_{s \in X} \mathcal{X}_{\{s\}} \times \mathcal{A}_{X \setminus \{s\}} = \bigcup_{s \in X} \{s\} \times \mathcal{A}_{X \setminus \{s\}}$$

$(\mathcal{X} * \mathcal{A})_X$ is all ways to pick out an element of X and put an $\mathcal{A}$-structure on everything else.

Powers and Rooted Structures

November-20-13 10:31 AM

Powers (k-tuples of structures)

If $\mathcal{A}$ is any class, $k \in \mathbb{N}$

$$\mathcal{A}^k := \begin{cases} \mathcal{A} * \mathcal{A} * \cdots * \mathcal{A}\ (k \text{ times}) & \text{if } k \geq 1 \\ \mathcal{E}_0 & \text{if } k = 0 \end{cases}$$

Notice: $\mathcal{A}^k * \mathcal{A}^l = \mathcal{A}^{k+l}$

$\mathcal{E}_0 * \mathcal{A} \equiv \mathcal{A}$

EGF for $\mathcal{A}^k$ is $A(x)^k$

Connected

A class of structures $\mathcal{A}$ is said to be connected if $\mathcal{A}_\emptyset = \emptyset$

Sequences/Tuples of Structures

If $\mathcal{A}$ is connected,

$\mathcal{A}^* := \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \mathcal{A}^3 \oplus \cdots$

EGF:

$$A(x)^0 + A(x)^1 + A(x)^2 + A(x)^3 + \cdots = \frac{1}{1 - A(x)}$$

Connectivity condition is necessary to make this defined, or could have infinitely many $\mathcal{A}_X^i$ nonzero.

Rooted Structures

If $\mathcal{A}$ is a class, define $\mathcal{A}^\bullet$ the class of rooted $\mathcal{A}$-structures

$\mathcal{A}_X^\bullet := \mathcal{A}_X \times X$

i.e. to make an $\mathcal{A}^\bullet$-structure on X: but an $\mathcal{A}$-structure on X and pick an element of X to call the root.

Note

This is not the same as $\mathcal{X} * \mathcal{A}$

Remark

$\mathcal{A}^\bullet \equiv \mathcal{A} \& (\mathcal{X} * \mathcal{E})$

$|\mathcal{A}_X^\bullet| = |\mathcal{A}_X| \cdot |X|$

$\therefore |\mathcal{A}_X^\bullet| = n|\mathcal{A}_X|$

$$A^\bullet(x) = \sum_{n=0}^{\infty} n|\mathcal{A}_n| \cdot \frac{x^n}{n!} = x\frac{d}{dx}A(x)$$

Assumption

From now on, we'll need the following assumption:

If $X \neq Y$ are two finite sets then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$

If not, we can fix it by replacing $\mathcal{A}_X \to \{X\} \times \mathcal{A}_X$

Interpretation

Class of structures $\leftrightarrow$ Ways of doing ____ to a set

$\mathcal{E} \to$ leave alone

$\mathcal{S} \to$ make a permutation

$\mathcal{E}_k \to$ filter: has k elements $\Rightarrow$ accept, otherwise reject

$\mathcal{X} * \mathcal{A}$ (where $\mathcal{X} = \mathcal{E}_1$)

$*$ means split into 2 sets

Then $\mathcal{X}$ filters first set — can only have 1 element.

$\mathcal{A}$ does whatever to second set.

Example

$\mathcal{E}^k \equiv \mathcal{F}(\cdot, N_k)$

e.g. $\mathcal{E}_X^2 \rightleftarrows \mathcal{F}(X, N_2)$

$$(S, X \setminus S) \to f: a \mapsto \begin{cases} 1 & \text{if } a \in S \\ 2 & \text{if } a \in X \setminus S \end{cases}$$

Example Connected Classes of Structures

- Connected graphs (taking connectivity to mean 1 component)
- $\mathcal{T}$ - trees
- $\mathcal{C}$ - cyclic permutations
- $\mathcal{E}^k$ for $k \geq 1$

  EGF: $\exp(x)^k = x^{kx} = \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = \sum_{n=0}^{\infty} k^n \cdot \frac{x^n}{n!}$

  $\left|(\mathcal{E}^k)_n\right| = k^n$, consistent with $|\mathcal{F}(N_n, N_k)|$
- $\mathcal{E}_{\geq 1} = \mathcal{E} \setminus \mathcal{E}_0 = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3 \oplus \cdots$

  Class of nonempty sets

Example: Class of Linear Orderings

$\mathcal{L} := \mathcal{X}^*$

$\mathcal{L}_X = \{(x_1, x_2, \ldots, x_n) \mid \{x_1, \ldots, x_n\} = X,\ x_i \neq x_j \text{ for } i \neq j\}$

- All possible orderings of the elements of X

EGF for $\mathcal{L}$

$$\frac{1}{1 - X(x)} = \frac{1}{1 - x}$$

Same as EGF for $\mathcal{S}$

Remark

$\mathcal{L}$ and $\mathcal{S}$ have the same EGF $\Rightarrow$ numerically equivalent.

However, they are not naturally equivalent.

Pictures for $\mathcal{S}$: Elements in disjoint cycles

Pictures for $\mathcal{L}$: A list of elements

You can't convert form one to the other as-is, requires an additional ordering of the elements of X to interpret a cycle as a list or vice versa.

Example Rooted Structures

$\mathcal{T}^\bullet$ = class of rooted trees.

Draw a tree then circle one node.

$\mathcal{T}^{\bullet\bullet}$ = class of dobuley rooted trees.

Draw a tree then circle one node and draw a box around another. They may be the same nodes.

Sets of Structures

November-22-13 10:35 AM

k-Sets of Structures

If $\mathcal{A}$ is a **connected** class define $\mathcal{E}_k[\mathcal{A}]$ as follows:
$\mathcal{E}_k[\mathcal{A}]_X = \{\{\alpha_1, \dots, \alpha_k\} \mid (a_1, \dots, a_k) \in \mathcal{A}^k\}$
(Error in notes, they omit connected)

Take an $\mathcal{A}^k$-structure and forget the order.

Therefore, an $\mathcal{E}_k[\mathcal{A}]$ structure on X is a k-set $\{a_1, .., a_k\}$ where $\alpha_i \in \mathcal{A}_{S_i}$ and $\{S_1, \dots, S_k\}$ is a set partition of X.

$$|\mathcal{E}_k[A]_X| = \frac{1}{k!}|\mathcal{A}_X^k|$$

Sets of Structures

If $\mathcal{A}$ is a connected class, define $\mathcal{E}[\mathcal{A}]$ the class of sets of $\mathcal{A}$-structures as
$\mathcal{E}[A] = \mathcal{E}_0[\mathcal{A}] \oplus \mathcal{E}_1[\mathcal{A}] \oplus \mathcal{E}_2[\mathcal{A}] \oplus \cdots$
Make a k-set of $\mathcal{A}$ structures for some $k \in \mathbb{N}$

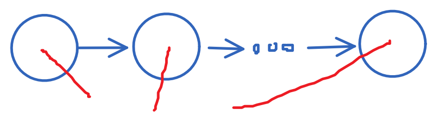
EGF for $\mathcal{E}[\mathcal{A}]$ is
$$\sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = \exp(A(x))$$

Recall $\mathcal{A}^k = \mathcal{A} * \mathcal{A} * \mathcal{A} * \cdots * \mathcal{A}$, k times

$$\mathcal{A}_X^k = \left\{(\alpha_1, \dots, \alpha_k) : \begin{array}{c} \alpha_i \text{ is a } \mathcal{A}\text{-structure on a subset } S_i \subseteq X \\ S_1 \cup \cdots \cup S_k = X \\ S_i \cap S_j = \emptyset, i \neq j \end{array}\right\}$$

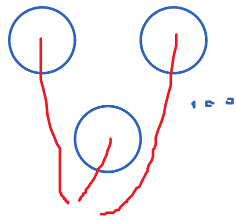
Pictures

$\mathcal{A}^k$:



Draw an $\mathcal{A}$ structure in each circle

$\mathcal{E}_k[\mathcal{A}]$



Draw an $\mathcal{A}$ structure in each circle

Examples

- $\mathcal{E}_k[\mathcal{X}] \equiv \mathcal{E}_k$
  - $\mathcal{E}_k[\mathcal{X}]$
    - Take set
    - partition it into k subsets
    - Ensure each of those subsets has exactly one element
  - $\mathcal{E}_k$
    - Make sure any set has k elements.
- $\mathcal{E}_1[\mathcal{A}] \equiv \mathcal{A}$ for any $\mathcal{A}$
- $\mathcal{E}_2[\mathcal{E}_2]_4 = \{\{12, 34\}, \{13, 24\}, \{14, 23\}\}$ where ij means $\{i, j\}$
- $\mathcal{E}_2[\mathcal{E}_2]_n = \emptyset$ for $n \neq 4$
- More generally, $\mathcal{E}_k[\mathcal{E}_2]$ is

  $$\mathcal{E}_k[\mathcal{E}_2]_X = \begin{cases} \text{The set of perfect matchings on the complete graph with vertex set } X & \text{if } |X| = 2k \\ \emptyset & \text{if } |X| \neq 2k \end{cases}$$

  k-sets of 2-sets

  \# of perfect matchings in K_{2k} is $(2k)!\,[x^{2k}]\frac{1}{k!}\left(\frac{x^2}{2!}\right)^k = \frac{(2k)!}{k!2^k}$
- $\mathcal{E}_k[\mathcal{E}_{\geq 1}]$ set partitions with k elements
  Recall $S(n, k) =$ # of set partitions of n-element set into k subsets.
  $S(n, k) = |\mathcal{E}_k[\mathcal{E}_{\geq 1}]_n|$
  EGF for $\mathcal{E}_k[\mathcal{E}_{\geq 1}] = \frac{1}{k!}(\exp(x) - 1)^k$
  $\therefore S(n, k) = n!\,[x^n]\frac{1}{k!}(\exp(x) - 1)^k$
  (This was in assignment 2)

Example Sets of Structures

Ptn is the class of set partitions.
$\text{Ptn}_X = \{\pi : \pi \text{ is a set parition of } X\}$
$\text{Ptn} \equiv \mathcal{E}[\mathcal{E}_{\geq 1}]$
EGF: $\exp(\exp(x) - 1)$
Total # of set partitions of an n-element subset is
$n!\,[x^n]\exp(\exp(x) - 1)$

- $\mathcal{E}[\mathcal{T}]$ is the class of forrests
- $\mathcal{E}[\mathcal{C}] \equiv \mathcal{S}$, Permutations are sets of cycles.
  $\therefore \exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x}$
  Recall, $C(x) = \log\left(\frac{1}{1-x}\right)$, $S(x) = \frac{1}{1-x}$
  So we have proved the combinatorial identity $\exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x}$

- Let $\mathcal{G}$ be the class of graphs. Let $\tilde{\mathcal{G}}$ be the class of connected graphs.
  $\mathcal{E}[\tilde{\mathcal{G}}] = \mathcal{G}$
  EGF for $\mathcal{G}$: $G(x) = \sum_{n=0}^{\infty} |\mathcal{G}_n| \frac{x^n}{n!} = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$
  EGF for $\tilde{\mathcal{G}}$: $\tilde{G}(x) = ???$
  $\exp\left(\tilde{G}(x)\right) = G(x)$
  $\tilde{G}(x) = \log(G(x)) = \log\left(\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}\right)$
  Don't forget $\log(A(x)) = \log\left(\frac{1}{1-\left(\frac{A-1}{A}\right)}\right) = -\log\left(\frac{1}{1-(1-G(x))}\right)$

- Let $\mathcal{J}$ be the class of (single undirected) 2-regular graphs.

i.e. every component is a cycle.
$\Rightarrow \mathcal{J} \equiv \mathcal{E}[\mathcal{H}]$ where $\mathcal{H}$ is the class of connected 2-regular graphs (cycles).
$\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \emptyset$ since a cycle consists of at least 3 vertices
$|\mathcal{H}_n| = \frac{(n-1)!}{2}$ for $n \geq 3$
$(n-1)!$ is number of directed cycles, 2 is the number of ways to direct an undirected cycle.

$\therefore$ EGF for $\mathcal{H}$ is
$$\mathcal{H}(x) = \sum_{n=3}^{\infty} \frac{(n-1)!}{2} \frac{x^n}{n!} = \frac{1}{2} \sum_{n=3}^{\infty} \frac{x^n}{n} = \frac{1}{2}\left(\log\left(\frac{1}{1-x}\right) - x - \frac{x^2}{2}\right)$$
EGF for $\mathcal{J}$ is
$$J(x) = \exp(H(x)) = \exp\left(\frac{1}{2}\left(\log\left(\frac{1}{1-x}\right) - x - x^2\right)\right) = \frac{\exp\left(-\frac{1}{2}x - \frac{1}{4}x^2\right)}{\sqrt{1-x}}$$

- $\mathcal{T}^{\bullet}$ = class of rooted trees
  $\mathcal{E}[\mathcal{T}^{\bullet}]$= class of forrests in which every tree is rooted.
  Claim: $\mathcal{X} * \mathcal{E}[\mathcal{T}^{\bullet}] \equiv \mathcal{T}^{\bullet}$
  Description of $\mathcal{X} * \mathcal{E}[\mathcal{T}^{\bullet}]$: pick a special vertex 'square' and put and $\mathcal{E}[\mathcal{T}^{\bullet}]$ structure on everything else.
  Bijection: Connect the special vertex 'square' to the root vertices of all the trees in $\mathcal{E}[\mathcal{T}^{\bullet}]$. Call 'square' the new root, then we have a single rooted tree.
  Let $T^{\bullet}(x)$ be the EGF for $\mathcal{T}^{\bullet}$.
  $\therefore x \exp(T^{\bullet}(x)) = T^{\bullet}(x)$
  This ist the right form to use $G(u) = \exp(u)$
  $$\therefore [x^n]T^{\bullet}(x) = \frac{1}{n}[u^{n-1}]\exp(u)^n = \frac{1}{n}[u^{n-1}]e^{un} = \frac{1}{n}[u^{n-1}]\sum_{k=0}^{\infty}\frac{n^k u^k}{k!} = \frac{1}{n} \cdot \frac{n^{n-1}}{(n-1)!}$$
  $$= \frac{n^{n-1}}{n!}$$
  $\therefore$ if $|X| = n$, the number of rooted trees with vertex set X is n^{n-1}.
  $\therefore$ The number of trees with vertex set X is n^{n-2}

Composition of Classes

November-25-13 11:05 AM

Composition of Classes

If $\mathcal{A}, \mathcal{B}$ are classes and $\mathcal{A}$ is connected, define $\mathcal{B}[\mathcal{A}]$

$$\mathcal{B}[\mathcal{A}]_X = \bigcup_{\xi \in \mathcal{E}[\mathcal{A}]_X} \{\xi\} \times \mathcal{B}_\xi$$

A $\mathcal{B}[\mathcal{A}]$ structure on X is a pair (ξ, β) where

- ξ is an $\mathcal{E}[\mathcal{A}]$ structure on X, hence $\xi = \{\alpha_1, \dots, \alpha_k\}$ where each α_i is an $\mathcal{A}$-structure on a subset $S_i \subseteq X$, and $\{S_1, \dots, S_k\}$ are a set partition of X
- β is a $\mathcal{B}$-structure on ξ.
    - What does this mean? ξ is a set of $\mathcal{A}$-structures, so we have a $\mathcal{B}$-structure where each label is an $\mathcal{A}$-structure, and the $\mathcal{A}$-structures taken together make up a $\mathcal{E}[\mathcal{A}]$ structure on X.

Pictorially

To draw a $\mathcal{B}[\mathcal{A}]$ structure:

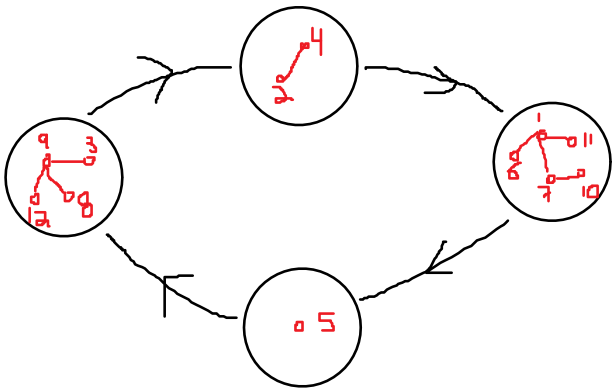
- Draw an unlabelled $\mathcal{B}$-structure where labelled objects are replaced by BIG circles.
- Inside

Theorem

The EGF for $\mathcal{B}[\mathcal{A}]$ is $B(A(x))$

Example

A $\mathcal{C}[\mathcal{T}]$ structure on N_{12}



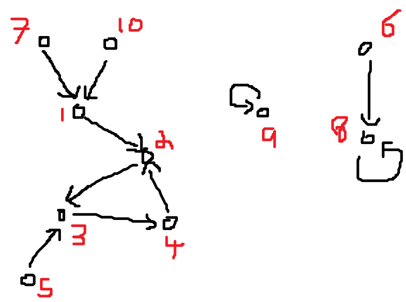
Examples

- Consistent with our earlier definitions of $\mathcal{E}[\mathcal{A}]$ and $\mathcal{E}_k[\mathcal{A}]$
- $\mathcal{X}^k[\mathcal{A}] \equiv \mathcal{A}^k$
- $\mathcal{L}[\mathcal{A}] = \mathcal{A}^*, \qquad \mathcal{L} = \mathcal{X}^*$
- $\mathcal{F}$: class of endofunctions

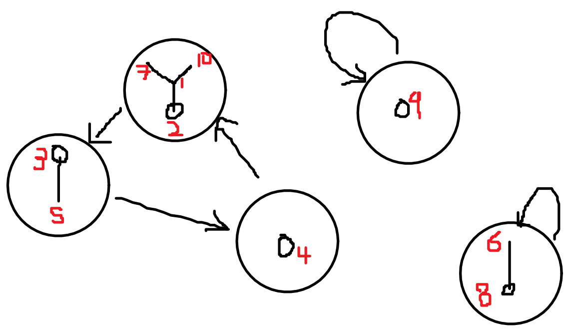
  $\mathcal{F}_X = \mathcal{F}(X, X)$

  Claim: $\mathcal{F} \equiv \mathcal{S}[\mathcal{T}^\bullet]$

  An $\mathcal{F}$-structure on N_{10}:



  Equivalent $\mathcal{S}[\mathcal{T}^\bullet]$-structure:



Proof of Theorem

$$|\mathcal{B}[\mathcal{A}]_n| = \left| \bigcup_{\xi \in \mathcal{E}[\mathcal{A}]_n} \{\xi\} \times \mathcal{B}_\xi \right| = \left| \bigcup_{k=0}^{\infty} \bigcup_{\xi \in \mathcal{E}_k[\mathcal{A}]_n} \{\xi\} \times \mathcal{B}_\xi \right| = \sum_{k=0}^{\infty} \sum_{\xi \in \mathcal{E}_k[\mathcal{A}]_n} 1 \cdot |\mathcal{B}_k|$$

$$= \sum_{k=0}^{\infty} |\mathcal{E}_k[\mathcal{A}]_n| \cdot |\mathcal{B}_k|$$

Note: EGF for

$$\mathcal{E}_k[\mathcal{A}] = \sum_{n=0}^{\infty} |\mathcal{E}_k[\mathcal{A}]_n| \frac{x^n}{n!} = \frac{A(x)^k}{k!}$$

EGF for $\mathcal{B}[\mathcal{A}]$ is

$$\mathcal{B}[\mathcal{A}] = \sum_{n=0}^{\infty} |\mathcal{B}[\mathcal{A}]_n| \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\mathcal{E}_k[\mathcal{A}]_n| \cdot |\mathcal{B}_k| \cdot \frac{x^n}{n!}$$

$$= \sum_{k=0}^{\infty} |\mathcal{B}_k| (\sum_{k=0}^{\infty} |\mathcal{E}_k[\mathcal{A}]_n| \cdot \frac{x^n}{n!} = \sum_{k=0}^{\infty} |\mathcal{B}_k| \cdot \frac{A(x)^k}{x!} = \mathcal{B}(A(x))$$

■

Example

Last time I showed you $\mathcal{F} \equiv \mathcal{S}[\mathcal{T}^\bullet]$

EGF for $\mathcal{F}$: $F(x) = \sum_{n=0}^{\infty} n^n \cdot \frac{x^n}{n!}$

EGF for $\mathcal{S}$: $S(x) = \dfrac{1}{1-x}$

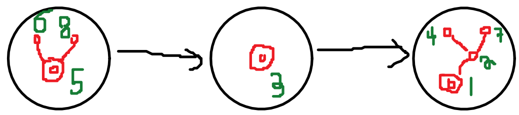
EGF for $\mathcal{T}^\bullet$: $T^\bullet(x) = \sum_{n=1}^{\infty} n^{n-1} \cdot \dfrac{x^n}{n!}$

Get a curious identity:

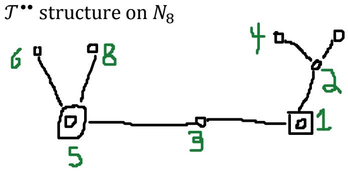
$$\sum_{n=0}^{\infty} n^n \cdot \frac{x^n}{n!} = \frac{1}{1 - \sum_{n=1}^{\infty} n^{n-1} \cdot \frac{x^n}{n!}}$$

Example: $\mathcal{E}_0 \oplus \mathcal{T}^{\bullet\bullet} \equiv \mathcal{L}[\mathcal{T}^\bullet]$

Proof by picture: An $\mathcal{L}[\mathcal{T}^\bullet]$ structure on N_8



A $\mathcal{T}^{\bullet\bullet}$ structure on N_8



Mixed Generating Functions

November-27-13 11:01 AM

Mixed = part ordinary, part exponential

Weight Function on Class of Structures

An $\mathbb{N}^r$-valued weight function ω on a class of structures $\mathcal{A}$ is a rule that assigns to each finite set X a weight function $\omega_X: \mathcal{A}_X \to \mathbb{N}^r$ satisfying the condition:

- If $X \rightleftarrows Y$ then there is a weight perserving bijection $\mathcal{A}_X \rightleftarrows \mathcal{A}_Y$
  i.e. if $|X| = |Y|$, then $\Phi_{\mathcal{A}_X}^{\omega_X}(y) = \Phi_{\mathcal{A}_Y}^{\omega_Y}(y)$

Mixed Generating Function

Given an a weight function ω on $\mathcal{A}$, define the mixed generating function:

$$A(x,y) = \sum_{n=0}^{\infty} \sum_{\sigma \in \mathcal{A}_n} y^{\omega(\sigma)} \cdot \frac{x^n}{n!}$$

Abuse of Notation

Often drop the X from ω_X and just write $\omega(\cdot)$ since (often? always?) ω can be described without reference to X.

Two ways to think about this:

1) As a generalization of EGF:
$$A(x,y) = \sum_{n=0}^{\infty} \Phi_{A_n}^{\omega}(y) \cdot \frac{x^n}{n!}$$
2) As a generalization of OGF.
   For all $\alpha \in \mathbb{N}^*$, define $\mathcal{A}^{[\alpha]}$:
   $\mathcal{A}_X^{[\alpha]} = \{\sigma \in \mathcal{A}_X : \omega(\sigma) = \alpha\}$
   $\mathcal{A}^{[\alpha]}$ is a class, because $\mathcal{A}_X \rightleftarrows \mathcal{A}_Y$ is weight preserving bijection $\Rightarrow$
   $\mathcal{A}_X^{[\alpha]} \rightleftarrows \mathcal{A}_Y^{[\alpha]}\ \forall \alpha$
$$A(x,y) = \sum_{n=0}^{\infty} \left(\sum_{\alpha \in \mathbb{N}^r} \left|\mathcal{A}_n^{[\alpha]} \cdot y^{\alpha}\right| \cdot \frac{x^n}{n!} \right) = \sum_{\alpha \in \mathbb{N}^r} y^{\alpha} \sum_{n=0}^{\infty} \left|\mathcal{A}_n^{[\alpha]}\right| \cdot \frac{x^n}{n!}$$
$$= \sum_{\alpha \in \mathbb{N}^r} y^{\alpha} A^{[\alpha]}(x)$$

Theorem

If ω is an $\mathbb{N}$-valued weight function on $\mathcal{A}$ and X is a finite set, $|X| = n$.
Then the average value of ω taken over all $\mathcal{A}_X$-structures is

$$\frac{[x^n] \frac{\partial}{\partial y} A(x,y)\Big|_{y=1}}{[x^n] A(x)}$$

Proof

☐ Exercise

Examples

- A weight function on $\mathcal{G}$:
  $\omega_X(\mathcal{G})$ = number of components for G, for $G \in \mathcal{G}_X$
- A weight function on $\mathcal{T}$:
  $\omega_X(\mathcal{T})$ = number of leaves of T, for $T \in \mathcal{T}_X$
- A weight function on Ptn
  $\omega_X(\pi) = |\pi|$ = number of subsets in π for $\pi \in Ptn_X$
- A weight function on $\mathcal{F}$:
  $\omega_X(\phi) = |\{v \in X \mid \phi(v) = v\}|, \quad \phi \in \mathcal{F}_X$
- A weight function on $\mathcal{S}$:
  $\omega_X(\sigma)$ = number of cycles of σ, $\sigma \in \mathcal{S}_X$

What do these have in common? ω_X can be read off from the picture, ignoring labels.

Example

For the class Ptn of set partitions, define the weight function $\omega(\pi) = |\pi|$
Compute the mixed generating function

$$\text{Ptn}(x,y) = \sum_{n=0}^{\infty} \sum_{\sigma \in Ptn_n} y^{\omega(\pi)} \frac{x^n}{n!}$$

Solution

Let $Ptn^{[k]}$ be the class of set partitions with k subsets.
As we saw before, $Ptn^{[k]} \equiv \mathcal{E}_k[\mathcal{E}_{\geq 1}]$

$$\therefore \text{Ptn}^{[k]}(x) = \frac{(\exp(x) - 1)^k}{k!}$$

$$\therefore \text{Ptn}(x,y) = \sum_{k=0}^{\infty} y^k \text{Ptn}^{[k]}(x) = \sum_{k=0}^{\infty} \frac{y^k (\exp(x) - 1)^k}{k!} = \exp\big(y(\exp(x) - 1)\big)$$

Jacobi Triple Product

December-02-13 10:31 AM

Jacobi Triple Product Formula

$$\sum_{h=-\infty}^{\infty} x^{h^2} y^h = \prod_{j=1}^{\infty} (1 + x^{2j-1} y)(1 + x^{2j-1} y^{-1})(1 - x^{2j})$$

Tree Comparison

All the followed have the same tree structure, but contain different extra information.

Binary Rooted Trees

Each child is labelled left or right.

Plane Planted Trees where each node has ≤ 2 children

Children of nodes are ordered. Same as BRT if 0 or 2 children, but if there is only 1 node then it is just '1st' not left or right.

(Labelled) Rooted Trees with at most 2 children for each node

No left/right or order. Just a tree with no additional information. (Has labels, but this does not change things.)

Commonalities

- Root
- Concept of children
- At most 2 children

Proof of Jacobi Triple Product Formula

Rewrite

$$\sum_{h=-\infty}^{\infty} x^{h^2} y^h \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j}} = \prod_{j=1}^{\infty} (1 + x^{2j-1} y)(1 + x^{2j-1} y^{-1})$$

Let $\mathcal{F}$ the the set of all subsets $A \subseteq \mathbb{Z}_{\text{odd}}$ (traditionally $\mathbb{Z} + \frac{1}{2} = \left\{\dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\}$) such that:

- There are finitely many negative odd integers in A
- There are finitely many positive odd integers not in A

Example

$A = \{-11, -7, -3, 3, 11, 13, 15, 17, 19, 21, \dots\}$

Includes -3, -7, -11, excludes 1, 5, 7, 9

We'll biject $\mathcal{F}$ with two different sets.

Method 1

Pairs of finite sets of odd positive integers

For $A \in \mathcal{F}$, let $\alpha = \{ j \mid j \in \mathbb{Z}_{\text{odd}},\ j > 0, j \notin A\}$, $\beta = \{-j \mid j \in \mathbb{Z}_{\text{odd}},\ j < 0,\ j \in A\}$

Example

For A given above, $\alpha = \{1,5,7,9\}$ and $\beta = \{5, 7, 11\}$

Define $\text{Energy}(\alpha, \beta) = \text{sum}(\alpha) + \text{sum}(\beta)$

$\text{Charge}(\alpha, \beta) = |\alpha| - |\beta|$

$$\sum_{\alpha,\beta} x^{\text{energy}(\alpha,\beta)} y^{\text{charge}(\alpha,\beta)} = \left(\sum_{\alpha} x^{\text{sum}(\alpha)} y^{|\alpha|}\right)\left(\sum_{\beta} x^{\text{sum}(\beta)} (y^{-1})^{|\beta|}\right)$$

Think of α, β as parititons with distinct parts, only odd parts

$$= \prod_{j=1}^{\infty} (1 + x^{2j-1} y)(1 + x^{2j-1} y^{-1})$$

Method 2

Charged Partitions = Lattice path of a certain type = partition of a number

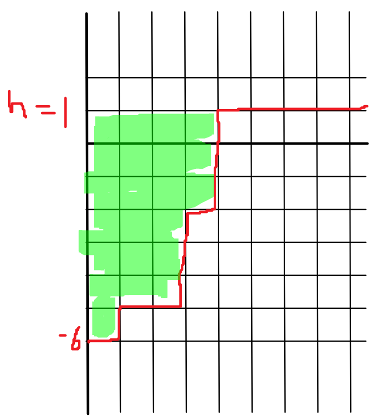
Start a lattice path at $(0, k)$ where $2k + 1$ is the smallest number of A.

For $i \in \mathbb{N}$,

$$\text{let } \sigma_i = \begin{cases} E & \text{if } 2(k + i) + 1 \in A \\ N & \text{otherwise} \end{cases}$$

Example

For A given above, $k = -6$



Get a Ferrers diagram bounded by the extension of this ray to the left of the y-axis, and the path.

Let λ be the associated partition. Let $h \in \mathbb{Z}$ be the number such that $(0, h)$ is the upper left corner.

Not hard to see you can get all possible partitions and all possible numbers. $A \leftrightarrow \mathbb{Z} \times \mathcal{Y}$

Claim

If $A \leftrightarrow (\alpha, \beta)$

Then $\text{energy}(\alpha, \beta) = 2|\lambda| + h^2,\ \text{charge}(\alpha, \beta) = h$

Proof

By induction on $\text{energy}(\alpha, \beta)$

Base case: $A = \{2k+1, 2k+3, 2k+5, \dots\}$

Induction Step

…

It follows that

$$\text{RHS} = \sum_{\alpha,\beta} x^{\text{energy}(\alpha,\beta)} y^{\text{charge}(\alpha,\beta)} = \sum_{(\alpha,\beta)} x^{2h(\lambda)+h^2} + y^h = \left(\sum_{h\in\mathbb{Z}} x^{h^2} y^h\right)\left(\sum_{\lambda\in\mathcal{Y}} x^{2h(\lambda)}\right)$$
$$= \text{LHS}$$