Optimization Intro

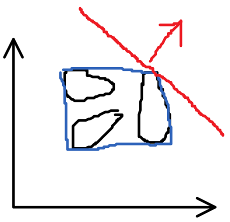
September-10-12 3:03 PM

Diophantine Equation

A Diophantine equation is an equation $p(x_1, \ldots, x_n) = 0$ where p is a polynomial with integer coefficients.

Problem (Hilbert's 10th)

Can we decide whether or not there exist $x_1, x_2, \ldots, x_n \in \mathbb{Z}$ such that $p(x_1, \ldots, x_n) = 0$ (given p)?
No. Not even if we fix $n = 9$



Optimization

Given a set S (the feasible region) and a function $f: S \to \mathbb{R}$ (the objective function) solve
$\max(f(x): x \in S)$ or $\min(f(x): x \in S)$
Note: $\min(f(x): x \in S) = -\max(-f(x): x \in S)$

Linear Programming

$f(x) = c^T x$ where $S(x) = \{x \in \mathbb{R}^n : Ax \leq b\}$
$c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$

Integer Linear Programming Problems

$f(x) = c^T x, \qquad S(x) = \{x \in \mathbb{Z}^n : Ax \leq b\}$

Convex Optimization

$\min(f(x): x \in S)$
$S \subseteq \mathbb{R}^n$ is convex, meaning $\forall x, y \in S,\ \forall s \in [0,1]\ \ xs + y(1-s) \in S$
f convex

Remarks

Consider an optimization problem $\min(f(x): x \in S)$
We can assume without much loss of generality that

1. $S \subseteq \mathbb{R}^n$
2. f is linear

$\min(f(x): x \in S) = \min(z: z = f(x), x \in S)$
Now, the function is linear so the optimum lies on an edge of the set, so can replace S by its convex hull. Therefore, can assume that

3. S is convex

Examples

1. **A two player game.**

Given $A \in \mathbb{R}^{m \times n}$
Rose chooses a row i and Colin independently chooses a column j then Colin pays row \$$a_{ij}$

Example

$A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$
If Rose chooses 2 she gets ≥ 1, if she chooses she gets ≥ -2
If she chooses the two rows with equal probability, she expects $\geq \min\left(\frac{2+1}{2}, \frac{-2+5}{2}\right) = \frac{3}{2}$
Rose wants to solve

$$\max_{p \in \mathbb{R}^n} \min_{j \in \{1..n\}} \sum_{i=1}^{m} p_i a_{ij} \ \ \text{subject to} \sum_{i=1}^{m} p_i = 1, \qquad p_1, \ldots p_m \geq 0$$

Equivalently: Maximize z subject to

$$z \leq \sum_{i=1}^{m} p_i a_{ij}, \qquad for\ j \in \{1, \ldots, n\}$$

$p_1 + \cdots + p_m = 1, \qquad p_1, \ldots, p_m \geq 0$
This is a linear program

2. **Weighted bipartite matching**

Problem:

Given n jobs, n workers and a "utility" a_{ij} for worker j to complete job i. Find an assignment maximizing the total utility (i.e. the sum of the utilities)

Formulation

Maximize

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1\ \forall\ i \in \{1, \ldots, n\}$$

$$\sum_{i=1}^{n} x_{ij} = 1\ \forall j \in \{1, \ldots n\}$$

$x_{ij} \in \{0,1\}\ \forall\ i,j \in \{1, \ldots, n\}$
This is an integer programming formulation.

3. **3D Modelling**

Problem

Given $a \in \mathbb{R}^{n \times n \times n}$, a_{ijk} is the utility of job i being completed by worker j on machine k, find an "assignment" of maximum total utility.

Formulation

Maximize

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ijk} x_{ijk}$$

subject to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ijk} = 1 \ \ \forall k \in \{1, \dots, n\}$$
$$\sum_{i=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \ \forall j \in \{1, \dots, n\}$$
$$\sum_{j=1}^{n}\sum_{k=1}^{n} x_{ijk} = 1 \ \forall i \in \{1, \dots, n\}$$
$$0 \leq x_{ijk} \leq 1 \ \text{ integer } \forall i, j, k \in \{1, \dots, n\}$$

Remark

The 3D Matching Problem is NP-hard and hence integer linear programming is NP-hard.

4. Diophantine Equations

$$(P)\begin{cases} \text{Minimize} \\ \sin(\pi x)^2 + \sin(\pi y)^2 + \sin(\pi z)^2 \\ \text{subject to} \\ x^3 + y^3 - z^3 = 0 \\ x, y, z \geq 1 \end{cases}$$

Not that (P) has optimal value $= 0$ if and only if there are non-negative integers such that $x^3 + y^3 = z^3$

Problem (Hilbert's 10th)

Can we decide whether or not there exist $x_1, x_2, \dots, x_n \in \mathbb{Z}$ such that $p(x_1, \dots, x_n) = 0$ (given p)?
No. Not even if we fix $n = 9$

Formulation

$$(P)\begin{cases} \text{Minimize } \sum_{i=1}^{n} \sin(\pi x_i)^2 \\ \text{subject to } p(x_1, \dots, x_n) = 0 \end{cases}$$

This has optimal value 0 iff p has an integer root.

5. Distance from Feasibility

Problem

Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m,$ and $z \in \mathbb{R}^n$, how far is z from the feasible region $\{x \in \mathbb{R}^n : Ax \leq b\}$?

Formulation

$$(P)\begin{cases} \text{Minimize } \sum_{i=1}^{n} (x_i - z_i)^2 \\ \text{Subject to } Ax \leq b \end{cases}$$

This is a convex optimization problem

Linear Algebra Review

September-12-12 2:57 PM

Remark

Matrices do not have ordered rows and columns. $A \in \mathbb{F}^{X \times Y}$, $\mathbb{F}$ is a field and X, Y are finite sets.

Fundamental Theorem of Linear Algebra

For $A \in \mathbb{F}^{m \times n},\ b \in \mathbb{F}^m$, exactly one of the following holds:

1. There exists $x \in \mathbb{F}^n$ such that $Ax = b$
2. There exists $y \in \mathbb{F}^m$ such that $y^T A = \vec{0}$ and $y^T b = 1$

Solutions to Linear Systems

$A \in \mathbb{F}^{m \times n}$ with $rank(A) = m$ and let $b \in \mathbb{F}^m$
Let A_j denote the j^{th} column of A and for $\beta \subseteq \{1, \dots, n\}$, let $A_\beta = [A_j : j \in B]$ We call β a basis if $|\beta| = m$ and A is non-singular.
Note that if β is a basis, then there is a unique solution to

a) $Ax = b$
b) $x_j = 0, j \notin \beta$

Since we are left with a square invertible matrix when removing the j's

We call this the **basic solution** for β

It is the unique solution to $Ax = b,\ x_j = 0\ \ j \notin \beta$

Support

The support of $x \in \mathbb{F}^n$ is $supp(x) = \{i : x_i \neq 0\}$

Theorem

For $A \in \mathbb{F}^{m \times n}, b \in \mathbb{F}^n$, if $Ax = b$ has a solution, then it has a solution whose support has size $\leq rank(A)$

Note that $Ax = b$ can be solved in $O(mn\, rank(A))$ arithmetic operations. Is this efficient? What about the size of the solution?

The size of a solution

For $a \in \mathbb{Z}$, define $size(a) \leq \lceil \log_2(|a| + 1) \rceil + 1 \leq \log_2(|a|) + 2, for\ a \geq 1$
$size\left(\frac{a}{b}\right) = size(a) + size(b)$

Permutation Definition of Determinant

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} A_i, \sigma_i$$

Crammer's Rule

For a matrix equation
$Ax = b$
the solution is
$x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, \dots, n$
A_i is the matrix formed by replacing the i^{th} column of A by the column vector b.

Proof of Theorem

Let $A \in \mathbb{Z}^{n+n}, b \in \mathbb{Z}^n$ and let L be the size of the largest entry in A or b
Suppose that A is non-singular

$size(\det(A)) \leq size(n!\,(2^L)^n) \leq 2 + \log_2(n!\,(2^L)^n) \leq 2 + n(\log_2 n + L)$

Now consider $x = A^{-1}b$
By Crammers' Rule each entry of $\det(A)\,.\,A^{-1}$ a determinant of a submatrix of A, and hence, has size $\leq 2 + n(\log_2 n + L)$
So each entry of x has size $\leq n + (L+1)\big(2 + n(\log_2 n + L)\big)$

This is polynomially bounded in the size of A, b

Additionally, Gaussian Elimination can be performed carefully (by triangularizing without scaling) such that all intermediate values are small.

Systems of Inequalities

September-14-12 2:59 PM

Theorem (Farkas' Lemma; Theorem 2.7)

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
Exactly one of the following hold.
(1) There exists $x \in \mathbb{R}^n$ such that $Ax \le b$
(2) There exists $y \in \mathbb{R}^m$ such that $y \ge 0, y^T A = 0$, and $y^T b = -1$

Variable Elimination (Fourier-Motzkin)

Removing a variable from a system of linear inequalities

Rewrite the inequalities as
(1.1) $x_n \ge g_i(x_1, \dots, x_{n-1}),\ i \in A_1$
(1.2) $x_n \le g_i(x_1, \dots, x_{n-1}),\ i \in A_2$
(1.3) $0 \ge g_i(x_1, \dots, x_{n-1}),\ i \in A_3$
(A_1, A_2, A_3) partition $\{1, \dots, n\}$

Note that there is a solution if and only if there exist $x_1, \dots, x_n \in \mathbb{R}$ satisfying (1.3) such that
$\max_{i\in A_1}\left(g_i(x_1, \dots, x_{n-1})\right) \le \min_{i\in A_2}\left(g_i(x_1, \dots, x_{n-1})\right)$

Equivalently
(2.1) $g_i(x_1, \dots, x_{n-1}) \le g_j(x_1, \dots, x_{n-1})\ \forall\ i \in A_1, j \in A_2$
(2.2) $0 \ge g_i(x_1, \dots, x_{n-1})\ \forall\ i \in A_3$

Not that this is a system of linear inequalities in $n-1$ variables.

Other Forms of Farkas' Lemma

Theorem (2)

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$
Exactly one of the following hold:
(1) There exists $x \in \mathbb{R}^n$ such that $Ax = b, x \ge 0$
(2) There exists $y \in \mathbb{R}^n$ such that $y^T A \ge 0,\ \ y^T b = -1$

Theorem (3)

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$
Exactly one of the following hold
(1) There exists $x \in \mathbb{R}^n$ such that $Ax \le b, x \ge 0$
(2) There exists $y \in \mathbb{R}^m$ such that $y^T A \ge 0,\ y^T b = -1$, and $y \ge 0$

- [x] Proof: Exercise

Geometry

Suppose $A = [A_1, \dots, A_n] \in \mathbb{R}^{m\times n}$
Define $cone(A_1, \dots, A_n) = \{A_1 x_1 + \dots + A_n x_n : x \in \mathbb{R}^n, x \ge 0\}$
Problem: is $b \in cone(A_1, \dots, A_n)$?
Equivalently, does $(Ax = b, x \ge 0)$ have a solution?
By the theorem
$Ax = b, x \ge 0$ xor $\exists y\ s.t.\ y^T A \ge 0, y^T b = -1$
there exists $\alpha \in \mathbb{R}^m$ such that $\alpha^T A \ge 0$ and $\alpha^T b = -1$

Equivalently,
$\alpha^T A_1 \ge 0, \alpha^T A_2 \ge 0, \dots, \alpha^T A_n \ge 0, \alpha^T b = -1$

Equivalently,
$A_1, A_2, \dots, A_n$ are contained in the half-space $\{x \in \mathbb{R}^m : \alpha^T x \ge 0\}$ but b is not.

Equivalently,
$cone(A_1, \dots, A_n) \subseteq \{x \in \mathbb{R}^m : \alpha^T x \ge 0\}$ but $b \notin \{x \in \mathbb{R}^m : \alpha^T x \ge 0\}$

Theorem

$b \notin cone(A_1, \dots, A_n)$ iff there is a hyperplane separating b from the cone.

Separating Hyperplane Theorem

Let $S \subseteq \mathbb{R}^m$ be a closed convex set and $b \in \mathbb{R}^m$. If $b \notin S$ then there is a hyperplane separating b from S.
Prove later

Proof of Theorem

Easy part: (1) and (2) cannot both hold.
If $Ax \le b$ and $y \ge 0$ then $y^T Ax \le y^T b$
But if $y^T A = 0$, then $0 \le y^T b$

It remains to prove that if (1) does not hold then (2) does.

Restatement

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be linear for $i \in \{1, \dots, n\}$
Define $f_i(x) = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n - b_i$

If $f_i(x) \le 0, i \in \{1, \dots, n\}$ has no solution, then there exists $\alpha \in \mathbb{R}^n_+$ such that
$$\sum_{i=1}^{n} \alpha_i f_i = 1$$

Where $\mathbb{R}_+ = \{z \in \mathbb{R} : z \ge 0\}$

Proceed by Induction:
$n = 0$ is a trivial base case: LHS is 0
Assume that Farkas' Lemma holds for systems with $n-1$ variables.

Break it down into equations with $n-1$ variables: (2.1) and (2.2)
Assuming that $Ax \le B$ has no solution, so (2.1) and (2.2) don't either.
Then by the inductive assumption, there exist $\alpha \in \mathbb{R}_+^{A_1 \times A_2}$ and $\beta \in \mathbb{R}_+^{A_3}$ such that
$$\sum_{i\in A_1}\sum_{j\in A_2} \alpha_{ij}(g_i - g_j) + \sum_{k\in A_3} \beta_k g_k = 1$$

For $i \in \{1, \dots, m\}$ we define
$$\alpha_i = \begin{cases} \sum_{j\in A_2} \alpha_{ij} & : \quad i \in A_1 \\ \sum_{j\in A_1} \alpha_{ji} & : \quad i \in A_2 \\ b_i & : \quad i \in A_3 \end{cases}$$
Now,
$$f_i(x_1, \dots, x_n) = \begin{cases} g_i(x_1, \dots, x_{n-1}) - x_n & : \quad i \in A_1 \\ -g_i(x_1, \dots, x_{n-1}) + x_n & : \quad i \in A_2 \\ g_i(x_1, \dots, x_n) & : \quad i \in A_3 \end{cases}$$
Now $\alpha \ge 0$ and
$$\sum_{i=1}^{n} \alpha_i f_i = \sum_{i\in A_1}\left(\sum_{j\in A_2} a_{ij}\right)(g_i - x_n) + \sum_{i\in A_2}\left(\sum_{j\in A_1} a_{ji}\right)(x_n - g_i) + \sum_{k\in A_3} \beta_k g_k$$
$$= \sum_{i\in A_1}\left(\sum_{j\in A_2} a_{ij}(g_i - x_n) + a_{ij}(x_n - g_j)\right) + \sum_{k\in A_3} \beta_k g_k$$
$$= \sum_{i\in A_1}\sum_{j\in A_2} a_{ij}(g_i - g_j) + \sum_{k\in A_3} \beta_k g_k = 1$$
As required.
This proves Farkas' Lemma

Proof of Theorem (2)

$Ax = b, x \ge 0$ can be rewritten as
$Ax \le b, \quad -Ax \le -b, \quad -x \le 0$
Let $A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}$ and $b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$
So (1) is equivalent to
(1') There exists $x \in \mathbb{R}^n$ such that $A'x \le b'$

By the Farkas' Lemma, this is equivalent to:
(2') There do not exist $y_1, y_2, y_3 \in \mathbb{R}^m$ such that
$[y_1, y_2, y_3]^T A' = 0, \quad [y_1, y_2, y_3]^T b' = -1, \quad y_1, y_2, y_3 \ge 0$

That is,
$y_1^T A - y_2^T A - y_3 = 0, \quad y_1^T b - y_2^T b = -1, \quad y_1, y_2, y_3 \ge 0$

That is
$(y_1 - y_2)^T A = y_3, \quad (y_1 - y_2)^T b = -1, \quad y_1, y_2, y_3 \ge 0$
So $y = y_1 - y_2$

Note to Self

Proof works in reverse:
Assume 2 is false so there does not exist y such that $y^T A \ge 0$ and $y^T b = -1$
Which means y_1, y_2, y_3 do not exist, and by Farkas' Lemma, there must be a solution to $A'x \le b'$
A solution to $A'x \le b'$ provides a solution to $Ax = b,\ x \ge 0$. ■

Proof of Theorem (3)

$Ax \le b, x \ge 0$ can be rewritten as
$Ax \le b, \quad -x \le 0$
Let $A' = \begin{bmatrix} A \\ -I \end{bmatrix}$ and $b' = \begin{bmatrix} b \\ 0 \end{bmatrix}$
Then (1) is equivalent to

(1'') There exists $x \in \mathbb{R}^n$ such that $A'x \leq b'$

By the Farkas' Lemma, this is equivalent to:
(2'') There do not exist $y_1, y_2 \in \mathbb{R}^m$ such that
$$Ay_1 - y_2 = 0, \qquad y_1^T b = -1, \qquad y_1, y_2 \geq 0$$
Set $y = y_1$ then have $Ay \geq 0, y^T b = -1, y \geq 0$

Linear Programming

September-19-12 2:55 PM

Linear Program

A linear program (or LP) is a problem of the form $\max(c^T x: Ax \leq b)$ or $\min(c^T x: Ax \geq b)$
where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$.

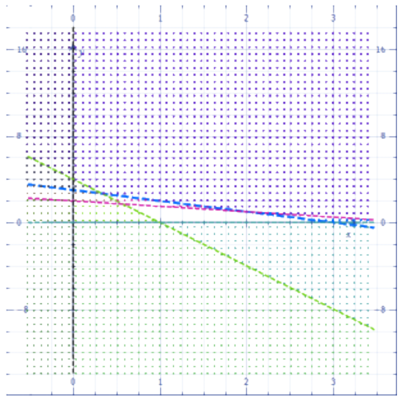
Note
$\max(c^T: Ax \leq b) = -\min(-c^T x: Ax \leq b)$

Remark

The problem of determining the best bound on the objective function via linear combination of constraints is a linear programming problem.

Example

$$(P)\begin{cases} \text{Maximize } x_2 \\ \text{subject to} \\ x_1 + x_2 \leq 3 \\ 4x_1 + x_2 \geq 4 \\ x_1 + 2x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{cases}$$



The maximum x^* satisfies $4x_1 + x_2 = 4$ and $x_1 + 2x_2 = 4$ so $x^* = \left[\frac{4}{7}, \frac{12}{7}\right]^T$ and the optimal value is $\frac{12}{7}$.

Problem

How in general can we prove that a given solution is optimal?
Equivalently, how can we generate upper-bounds on the value?

Answer

Take linear combinations of the constraints.

$x_1 + x_2 \leq 3 \quad \times 2$
$4x_1 + x_2 \geq 4 \quad \times -1$
$x_1 + 2x_2 \leq 4 \quad \times 2$

$5x_2 \leq 10 \Rightarrow x_2 \leq 2$
So each objective function has objective value ≤ 2
Note that to prove that x^* is optimal we shall only use inequalities that x^* satisfies with equality:
$4x_1 + x_2 \geq 4 \quad \times -1$
$x_1 + 2x_2 \leq 4 \quad \times 4$
$7x_2 \leq 12 \Rightarrow x_2 \leq \dfrac{12}{7}$
So x^* is optimal.

Remark

The problem of determining the best bound on the objective function via linear combination of constraints is a linear programming problem.

$x_1 + x_2 \leq 3$	$\times y_1 \geq 0$
$4x_1 + x_2 \geq 4$	$\times y_2 \leq 0$
$x_1 + 2x_2 \leq 4$	$\times y_3 \geq 0$

$x_1, x_2 \geq 0$

Take the linear combination:
$(y_1 + 4y_2 + y_3)x_1 + (y_1 + y_2 + 2y_3)x_2 \leq 3y_1 + 4y_2 + 4y_3$
$x_1, x_2 \geq 0$

We want
$0x_1 + 1x_2 \leq (y_1 + 4y_2 + y_3)x_1 + (y_1 + y_2 + 2y_3)x_2$
^
(objective function)

So we want $y_1 + 4y_2 + y_3 \geq 0$ and $y_1 + y_2 + 2y_3 \geq 1$

The dual of (P) is

$$(D)\begin{cases}\text{Minimize } 3y_1+4y_2+4y_3\\ \quad\text{subject to}\\ y_1+4y_2+y_3\geq 0\\ y_1+y_2+2y_3\geq 1\\ y_1\geq 0, y_2\leq 0, y_3\geq 0\end{cases}$$

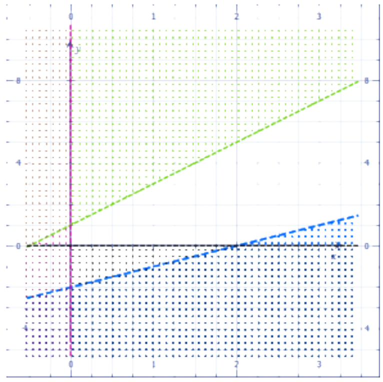
By construction, if x is feasible for (P) and y is feasible for (D) then $x_2 \leq 3y_1+4y_2+4y_3$

Note that for $x^* = \left[\frac{4}{7}, \frac{12}{7}\right]^T$ and $y^* = \left[0, -\frac{1}{7}, -\frac{4}{7}\right]$ we get equality.

Unboundedness

Example

$$(P)\begin{cases}\text{Maximize } x_1+x_2\\ \quad\text{subject to}\\ x_1-x_2\leq 2\\ -2x_1+x_2\leq 1\\ x_1,x_2\geq 0\end{cases}$$



Let $\hat{x} = \begin{bmatrix}1\\2\end{bmatrix}$ and $d = \begin{bmatrix}1\\1\end{bmatrix}$

Then $\hat{x}+\alpha d$ is feasible for all $\alpha \geq 0$ and has objective value $3+2\alpha$ so (P) is unbounded.

Note that the half-line $\{\hat{x}+\alpha d: \alpha \geq 0\}$ is contained in the feasible region and $c^T d > 0$

In general, this is not always possible for unbounded sets

$F = \left\{\begin{pmatrix}x\\y\end{pmatrix}: y \geq x^2\right\}, \qquad maximize\ \left\{x: \begin{pmatrix}x\\y\end{pmatrix} \in F\right\}$

Theorems of LP

September-21-12 3:02 PM

Theorem
"Fundamental Theorem of Linear Programming"

Every LP is either
(1) infeasible,
(2) unbounded, or
(3) has an optimal solution.

Lemma 1

Consider an LP
$(P)\ \max(c^T x: Ax \le b)$
$A \in \mathbb{R}^{m\times n}, b \in R^m, c \in \mathbb{R}^n$
Suppose that $\hat{x}$ is a feasible solution with $c^T\hat{x} = \gamma$

If the column A_1 is a linear combination of the other columns, then either
(1) (P) has a feasible solution $\tilde{x}$ with $c^T\tilde{x} = \gamma$ and $\tilde{x}_1 = 0$, or
(2) there exists $d \in \mathbb{R}^n$ such that $Ad = 0$ and $c^T d > 0$
(here (P) is unbounded)

As far as I can tell, this is not a xor.

Lemma 2

Let $A'x \le b'$ be the subsystem of $Ax \le b$ that $\hat{x}$ satisfies with equality.
Then $\hat{x}$ is an extreme point of $\{x \in \mathbb{R}^n: Ax \le b\}$ if and only if $rank(A') = n$

Consequence

Note that there are only finitely many extreme points of $\{x \in \mathbb{R}^n: Ax \le b\}$
For each subsystem $A'x \le b'$ of $Ax \le b$ with $rank(A') = n$ there is at most one solution to $A'x = b'$

Geometry

Let $z_1, .., z_k \in \mathbb{R}^n$.
We say that x is a **convex combination** of $z_1, \dots, z_k$ if there exist $t_1, \dots, t_k \in \mathbb{R}$ such that
$x = t_1 z_1 + \dots + t_k z_k$
$t_1 + \dots + t_k = 1, \qquad t \ge 0$
We define the **convex hull** of $\{z_1, \dots, z_k\}$, denoted $conv(z_1, \dots, x_n)$ to be the set of all convex combinations of $z_1, \dots, z_k$.

Claim

$conv(z_1, \dots, z_,)$ is the smallest convex set that contains $z_1, \dots z_k$

Theorem

Let $A \in \mathbb{R}^{m\times n}$ with $rank(A) = n$, and let $b \in \mathbb{R}^m$.
Let $P = \{x \in \mathbb{R}^n: Ax \le b\}$
Let $K = \{x \in \mathbb{R}^n: Ax \le 0\}$
and let C be the convex hull of the extreme points of P.
For each $x \in P$ there exist $z \in C$ and $d \in K$ such that $x = z + d$

Note

For each $z \in C$ and $d \in K$, $z + d \in P$

Corollary 1

Consider the LP
$(P)\ \max(c^T x: Ax \le b)$
where $A \in \mathbb{R}^{m\times n}$ with $rank(A) = n$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.
Either
1) (P) is infeasible
2) There is an extreme point of $\{x \in \mathbb{R}^n: Ax \le b\}$ that is optimal for (P), or
3) There is a feasible half-line $\{x + \lambda d: \lambda \ge 0\}$ with $c^T d > 0$.
(Hence (P) is unbounded)

Corollary 2

Consider the LP
$(P)\ \max(c^T x: Ax \le b)$
where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

If (P) feasible and bounded, then (P) has an optimal solution.

Corollary 3 (Unboundedness Theorem)

Consider the LP:
$(P)\ \max(c^T x: Ax \le b)$
Then (P) is unbounded if and only if there is a feasible half-line $\{\hat{x} + \alpha\hat{d}: \alpha \ge 0\}$ with $c^T d > 0$.

That theorem is not vacuous:
Consider

$$(NLP)\begin{cases} \text{Minimize} \dfrac{1}{x} \\ \text{subject to } x \ge 1 \end{cases}$$

Proof of Lemma 1

There exists $z \in \mathbb{R}^n$ such that $Az = 0$ and $z_1 = -1$
We may assume that $c^T z = 0$ since otherwise (2) holds with $d = z$ or $d = -z$.
Let $\tilde{x} = \hat{x} + \hat{x}_1 z$. Then $\tilde{x}_1 = 0$, $\hat{x}$ is feasible and $c^T\tilde{x} = c^T\hat{x} = \gamma$. ∎

Proof of Lemma 2

Suppose that $rank(A') = n$ and $\hat{x} = \lambda x^1 + (1-\lambda)x^2$ where $0 < \lambda < 1$ and x^1 and x^2 are feasible. $A'x^1 \le b'$, $A'x^2 \le b'$ and $A'\hat{x} = b$ we have $A'x^1 = b'$, $A'x^2 = b'$.
However, $rank(A') = n$ so $x^1 = x^2$ so $\hat{x}$ is an extreme point.

Conversely, suppose that $rank(A') < n$. Then there exists $d \in \mathbb{R}^n$ such that $A'd = 0$ and $d \ne 0$. For small $\epsilon > 0$, $\hat{x} - \epsilon d$, $\hat{x} + \epsilon d \in \{x \in \mathbb{R}^n: Ax \le b\}$
So $\hat{x}$ is not an extreme point.

Proof of Claim (Exercise)

Let S be any convex set containing $\{z_1, \dots, z_k\}$.
Let $x \in conv(z_1, \dots, z_k)$ such that $\sum_{i=1}^k t_i z_i = x$ and $\sum_{i=1}^k t_i = 1,\ t_i \ge 0$
Induct on j to show that

$$x_j = \frac{\sum_{i=1}^j t_i z_i}{\sum_{i=1}^j t_i} \in S$$

$j = 1$: $x_1 = z_1 \in S$
Assume $x_{j-1} \in S$ then

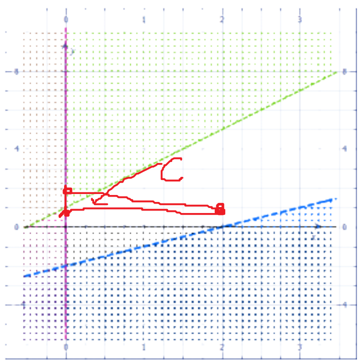
$$x_j = \frac{\sum_{j=1}^j t_i z_i}{\sum_{i=1}^j t_i} = \frac{\left(\sum_{i=1}^{j-1} t_i\right)}{\sum_{i=1}^j t_i} x_{j-1} + \frac{t_j}{\sum_{i=1}^j t_i} z_j \in S$$

by convexity of S.

Therefore, $x = x_k \in S$ so $conv(z_1, \dots, z_k) \subseteq S$.

Example of Theorem

$$(P) = \begin{cases} x_1 - x_2 \le 2 \\ -2x_1 + x_2 \le 1 \\ x_1, x_2 \ge 0 \end{cases}$$



$$(K) = \begin{cases} x_1 - x_2 \le 0 \\ -2x_1 + x_2 \le 0 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}3\\3\end{bmatrix}$$
$\in P \quad \in C \quad \in K$

Proof of Theorem

Let $\hat{x} \in$ P and let $A'x \le b'$ be the subsystem of $Ax \le b$ that $\hat{x}$ satisfies with equality.
We may assume that:
1. If $\tilde{x} \in$ P satisfies more of the constraints with equality than $\hat{x}$, then there exist $\tilde{z} \in$ C and $\tilde{d} \in$ K such that $\tilde{x} = \tilde{z} + \tilde{d}$.
   If Theorem did not hold, then take $\hat{x}$ to be the largest counter-example
2. $\hat{x}$ is not an extreme point. (Otherwise $\hat{z} = \hat{x}$, $\hat{d} = 0$)

By (2), $rank(A') < n$ (Lemma 2) so there exists $d \in \mathbb{R}^n$ such that $A'd = 0$ and $d \ne 0$.

Since $rank(A) = n$, $Ad \ne 0$
By possibly replacing d with $-d$ we may assume that Ad has a negative entry.

Case 1

$Ad \le 0$ (So $d \in K$)
Choose $t_1 = \max(t \in \mathbb{R}: x - td \in P\}$
Since Ad has a negative entry, this is well defined.

Let $x_1 = \hat{x} - t_1 d$
Now x_1 satisfies more of the inequalities $Ax \le b$ with equality than $\hat{x}$

Consider the LP:
$(P)\ \max(c^T x: Ax \leq b)$
Then (P) is unbounded if and only if there is a feasible half-line $\{\hat{x} + \alpha\hat{d}: \alpha \geq 0\}$ with $c^T d > 0$.

Since Ad has a negative entry, this is well defined.

Let $x_1 = \hat{x} - t_1 d$
Now x_1 satisfies more of the inequalities $Ax \leq b$ with equality than $\hat{x}$
So by (1), there exists $z_1 \in C$ and $d_1 \in K$ such that
$x_1 = z_1 + d_1$
Hence
$\hat{x} = x_1 + t_1 d = z_1 + (d_1 + t_1 d)$
Note that $z_1 \in C$ and $d_1 + t_1 d \in K$ as required.

Case 2
not Case 1
(That is, Ad has both positive and negative entries).
Let $t_1 = \max(t \in \mathbb{R}: \hat{x} - td \in P)$ and $t_2 = \max(t \in R: \hat{x} + td \in P)$
Note that these are well defined and positive
Let $x^1 = \hat{x} - t_1 d,\ x^2 = \hat{x} + t_2 d$

Note that x^1 and x^2 satisfy more constraints with equality than $\hat{x}$, there exist $z^1, z^2 \in C$ and $d^1, d^2 \in K$ such that $x^1 = z^1 + d^1$ and $x^2 = z^2 + d^2$
Now $\hat{x} = \frac{t_2}{t_1 + t_2}x^1 + \frac{t_1}{t_1 + t_2}x^2 = \frac{t_2}{t_1 + t_2}(z^1 + d^1) + \frac{t_1}{t_1 + t_2}(z^2 + d^2)$
$= \frac{1}{t_1 + t_2}(t_2 z^1 + t_1 z^2) + \frac{1}{t_1 + t_2}(t_2 d^1 + t_1 d^2)$
Since C and K are convex
$\frac{1}{t_1 + t_2}(t_2 z^1 + t_1 z^2) \in C$ and $\frac{1}{t_1 + t_2}(t_2 z^1 + t_1 z^2) \in K$
■

Proof of Corollary 1
Assume that (P) is feasible. Let γ be the maximum objective value of an extreme point of $\{x \in \mathbb{R}^n: Ax \leq b\}$

We may assume that there is a feasible solution $\hat{x}$ with $c^T x > \gamma$
(otherwise $\hat{x}$ satisfies (2))
By the Theorem, we can write $\hat{x} = \hat{z} + \hat{d}$ where
$\hat{d} \in \{x \in \mathbb{R}^n: Ax \leq 0\}$ and $\hat{z}$ is in the convex hull of the extreme points of $\{x \in \mathbb{R}^n: Ax \leq b\}$. Note that $c^T\hat{z} \leq \gamma$
Hence $c^T\hat{d} > 0$ and $\hat{x} + \lambda\hat{d}$ is feasible for all $\lambda \geq 0$. So (3) is satisfied.

Proof of Corollary 2
By Lemma 1 we may assume that $rank(A) = n$
((P) is feasible and bounded so if $rank(A) < n$ then can reduce problem to one in one fewer dimension)

Then Corollary 2 follows from the theorem.

Proof of Corollary 3
$\Leftarrow$ Easy
$\Rightarrow$ By lemma 1, we may assume that $rank(A) = n$
(Suppose $rank(A) < n$, then either get the ray automatically, or can induct with one fewer column in A)

Now the result is an immediate corollary of Corollary 1.

Polytopes

September-28-12 2:43 PM

Polytope

A set of the form $\{x \in \mathbb{R}^n : Ax \le b\}$ is called a polyhedron.
A bounded polyhedron is a polytope.

Corollary 4

Every polytope is the convex hull of its extreme points.

Proof

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$

Corollary 5

For $z_1, \dots, z_t \in \mathbb{R}^n$, $conv(z_1, \dots, z_t)$ is a polytope.

Valid Inequality

We call an inequality $\alpha^T x \le \beta$ **valid** for $conv(z_1, \dots, z_t)$ if $\alpha^T z_i \le \beta$ for each $i \in \{1, \dots, t\}$

Lemma 1

If $\hat{x} \in \mathbb{R}^n$ is not contained in $conv(z_1, \dots, z_t)$, then there is a valid inequality such that $\alpha^T \hat{x} > \beta$

(See Homework Problems page)

Corollary 6

A set $S \subseteq \mathbb{R}^n$ is a polytope if and only if it is the convex hull of a finite set of points.

Carathéodory's Theorem

Let $S \subseteq \mathbb{R}^n$ be finite. Then any point in $conv(S)$ can be written as a convex combination of at most $n+1$ points in S.

Proof: assignment 2 (or see MATH 245)

Theorem

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$
If the system $Ax \le b$ is infeasible then it contains an infeasible subsystem with at most $n+1$ inequalities.

Proof: assignment 2

Equivalently

IF $H_1, \dots, H_m \subseteq \mathbb{R}^n$ are halfspaces with empty intersection (that is, $H_1 \cap \dots \cap H_m = \emptyset$), then some subcollection of at most $n+1$ of these halfspaces has an empty intersection.

Corollary

If $P_1, \dots, P_m \subseteq \mathbb{R}^n$ are polyhedra with empty intersection, then some subcollection of $\le n+1$ of these polyhedra has an empty intersection.

Proof

Consider all inequalities defining these polyhedra. A subset of $\le n+1$ of them are infeasible, so use the at most $n+1$ polyhedra those inequalities come from.

Helly's Theorem

If $S_1, \dots, S_m \subseteq \mathbb{R}^n$ are convex sets with empty intersection then then there is a subcollection of $\le n+1$ has empty intersection.

Proof of Corollary 4

Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a polytope. Since P is bounded, it does not contain a line so $rank(A) = n$ (See assignment 1)

Assume feasible (if infeasible get Corollary automatically). Not contains a line $\Rightarrow$ contains an extreme point $\hat{x} \Rightarrow A'\hat{x} = b'$, $rank(A') = n$. $n \ge rank(A) \ge rank(A') = n \Rightarrow rank(A) = n$

By the theorem, if P is not in the convex hull of its extreme points, then there exists $\hat{x} \in P$ that can be written as $\hat{z} + \hat{d}$ where $\hat{z}$ is in the convex hull of the extreme points and $\hat{d} \in \{x \in \mathbb{R}^n : Ax \le 0\}$ with $\hat{d} \ne 0$.
Then $\{\hat{x} + \alpha\hat{d} : \alpha \ge 0\}$ is contained in P, contradicting the statement that P is bounded. ■

Proof of Corollary 5

$Q_0 = \left\{\begin{bmatrix}\alpha \\ \beta\end{bmatrix} :\ \alpha^T z_1 \le \beta, \dots, \alpha^T z_t \le \beta\right\}$

Note that

1) This is a cone (since you can scale valid inequalities by non-negative numbers)
2) Q_0 is a polyhedron since it is defined by a finite set of linear inequalities.

Now define

$Q_1 = \left\{\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \in Q_0 : -\underline{1} \le \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \le \underline{1}\right\}$

Now Q_1 is a polytope. Let $\begin{bmatrix}\alpha^1 \\ \beta^1\end{bmatrix}, \dots, \begin{bmatrix}\alpha^s \\ \beta^s\end{bmatrix}$ be the extreme points of Q_1.

Let $P = \{x \in \mathbb{R}^n : (\alpha^1)^T x \le \beta^1 \dots (\alpha^s)^T x \le \beta^s\}$

Claim

$P = conv(z_1, \dots, z_t)$

Proof

All inequalities in definition of P are valid, so $z_1, \dots, z_t \in P$ so $conv(z_1, \dots, z_t) \subseteq P$

Suppose that $P \ne conv(z_1, \dots, z_t)$
Then there exists $\tilde{x} \in P - conv(z_1, \dots, z_t)$
By Lemma 1, then there exists $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \in Q_0$ such that $\alpha^T \tilde{x} > \beta$.
By scaling we may assume that $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \in Q_1$
By Corollary 4, $\exists\, \lambda_1, \dots, \lambda_s \ge 0$ with $\lambda_1 + \dots + \lambda_s = 1$ and $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \lambda_1 \begin{bmatrix}\alpha^1 \\ \beta^1\end{bmatrix} + \dots + \lambda_s \begin{bmatrix}\alpha^s \\ \beta^s\end{bmatrix}$
Now $\beta < \alpha^T \tilde{x} = \lambda_1 (\alpha^1)^T \tilde{x} + \dots + \lambda_s (\alpha^s)^T \tilde{x} \le \lambda_1 \beta^1 + \dots \lambda_s \beta^s = \beta$. Contradiction. ■

Example Proof of Carathéodory's Theorem (n=2)

Split polygon into triangles, each point is in one of the triangles and can be written as combination of three points of triangle.

In higher dimensions, pick a vertex. Project interior points to opposite face, which has dimension $n-1$. Induct.

Proof of Helly's Theorem

We may assume that $m \ge n+1$
Suppose that each subcollection of $n+1$ of the sets has nonempty intersection. Then there is a set $X \subseteq \mathbb{R}^n$ with $|X| \le \binom{m}{n+1}$ so that each subcollection of $n+1$ of the sets contains an element of X.

For $i \in \{1, \dots, m\}$ define $P_i = conv(X \cap S)$, a polytope (by Corollary 6)
Every $n+1$ of these polytopes has non-empty intersection so
$\bigcap_{i=1}^{m} P_i \ne \emptyset$ so $\bigcap_{i=1}^{m} S_i \supseteq \bigcap_{i=1}^{m} P_i \ne \emptyset$ contradiction ■

Duality

October-03-12 2:30 PM

Duality

Consider the LP

$$(P)\begin{cases}\text{maximize} & c^Tx\\ \text{subject to} & Ax\le b\end{cases}$$

If $y\in\mathbb{R}^m$ and $y\ge 0$ then $y^TAx\le y^Tb$ is a valid inequality for (P)

If $y^TA=c^T$, then $c^Tx\le y^Tb$ gives you a bound on the objective value

Dual

The **dual** of (P) is the linear program

$$(D)\begin{cases}\text{minimize} & b^Ty\\ \text{subject to} & A^Ty=c\\ & y\ge 0\end{cases}$$

Weak Duality Theorem

If $x\in\mathbb{R}^n$ is feasible for (P) and $y\in\mathbb{R}^m$ is feasible for (D), then $c^Tx\le b^Ty$

Corollary 1

If (P) is unbounded then (D) is infeasible.

Corollary 2

If (D) is unbounded then (P) is infeasible.

Corollary 3

If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D) and $c^T\bar{x}=b^T\bar{y}$, then $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D)

Strong Duality Theorem

If (P) has an optimal solution $\bar{x}$ then (D) has an optimal solution $\bar{y}$, and $c^T\bar{x}=b^T\bar{y}$.

Relationship between Primary and Dual

(P) \ (D)	infeasible	unbounded	optimal
infeasible	Yes - exercise	Yes	No - exercise
unbounded	Yes	No	No
optimal	No	No	Yes

Alternate Forms

Consider the following LPs

$$(P1)\begin{cases}\text{maximize} & c^Tx\\ \text{subject to} & Ax\le b\end{cases}$$

$$(P2)\begin{cases}\text{maximize} & c^T(x^1-x^2)\\ \text{subject to} & A(x^1-x^2)\le b\\ & x^1,x^2\ge 0\end{cases}$$

$$(P3)\begin{cases}\text{maximize} & c^T(x^1-x^2)\\ \text{subject to} & A(x^1-x^2)+s=b\\ & x_1,x_2,s\ge 0\end{cases}$$

Claim

For any $\gamma\in\mathbb{R}$, the following are equivalent

1. $(P1)$ has a feasible solution with objective value γ
2. $(P2)$ has a feasible solution with objective value γ
3. $(P3)$ has a feasible solution with objective value γ

Standard Inequality Form

$(P2)$ is in standard inequality form

$$(PSI)\begin{cases}\max c^Tx\\ \text{subject to}\\ Ax\le b\\ x\ge 0\end{cases}$$

The dual of (PSI) is

$$(DSI)\begin{cases}\min b^Ty\\ \text{subject to}\\ A^Ty\ge c\\ y\ge 0\end{cases}$$

Standard Equality Form

$(P3)$ is in standard equality form

$$\begin{cases}\max c^Tx\end{cases}$$

Proof of Weak Duality Theorem

$c^Tx=(y^TA)x=y^T(Ax)\le y^Tb=b^Ty$ ■

Proof of Corollary 1

Contrapositive is obvious

Proof of Strong Duality Theorem

Consider the system

$$(1)\begin{cases}-c^Tx+b^Ty & \le 0\\ Ax & \le b\\ -A^Ty & =-c\\ y & \ge 0\end{cases}$$

If $\bar{x},\bar{y}$ satisfy (1) then $\bar{x}$ is feasible for (P), $\bar{y}$ is feasible for (D) so by the weak duality theorem, $c^T\bar{x}=b^T\bar{y}$. So $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D) as required.

So we may assume that (1) has no solution.

Multiply the rows of (1):

$$(1)\begin{cases}-c^Tx+ & b^Ty & \le 0 & \bar{z}\ge 0\\ Ax & & \le b & \bar{y}\ge 0\\ & -A^Ty & =-c & \bar{x}\\ & y & \ge 0\end{cases}$$

Claim

If (1) has no solution then there exist $\bar{x}\in\mathbb{R}^n,\bar{y}\in\mathbb{R}^m$ and $\bar{z}\in\mathbb{R}$ satisfying

$$(2)\begin{cases}-c^T\bar{x}+ & b^T\bar{y} & \le 0\\ A\bar{x} & & \le\bar{z}b\\ & A^T\bar{y} & =\bar{z}c\\ & \bar{y} & \ge 0\\ & \bar{z} & \ge 0\end{cases}$$

Proof:

Exercise (Use Farkas' Lemma)

Let $A'=\begin{bmatrix}-c^T & b^T\\ A & 0\\ 0 & A^T\\ 0 & -A^T\\ 0 & -I\end{bmatrix}$ and $b'=\begin{bmatrix}0\\ b\\ c\\ -c\\ 0\end{bmatrix}$

Then (1) can be written as

$A'\begin{bmatrix}x\\ y\end{bmatrix}\le b'$

Since we assome no solution to (1) exists, by Farkas' lemma there must exist $z\ge 0$ such that $A'^Tz=0$ and $b'^Tz=-1$

Write $z=\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{bmatrix}$ for $z_1\in\mathbb{R},z_2,z_5\in\mathbb{R}^m,z_3,z_4\in\mathbb{R}^n$

then

$A'^Tz=\begin{bmatrix}-cz_1+A^Tz_2\\ bz_1+Az_3-Az_4-z_5\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$b'^Tz=b^Tz_2+c^Tz_3-c^Tz_4=-1$

Let $\bar{z}=z_1$, $\bar{y}=z_2$, and $\bar{x}=z_4-z_3$ then

$-\bar{z}c+A^T\bar{y}=0\Rightarrow A^T\bar{y}=\bar{z}c$

$\bar{z}b-A\bar{x}-z_5=0\Rightarrow A\bar{x}=\bar{z}b-x_5\le\bar{z}b$

$-c^T\bar{x}+b^T\bar{y}=-1\le 0$ ■

Consider a solution $(\bar{x},\bar{y},\bar{z})$ to (2)

Case 1: $\bar{z}>0$

We can scale $(\bar{x},\bar{y},\bar{z})$ so that $\bar{z}=1$. Now $(\bar{x},\bar{y})$ satisfies (1) — contradiction.

Case 2: $\bar{z}=0$

Now $\bar{y}^TA=0$ and $\bar{y}\ge 0$. Since (P) is feasible $\bar{y}^Tb\ge 0$

($A^T\bar{y}=0$ & $\bar{y}^Tb<0$ is proof of infeasibility)

That is $b^T\bar{y}\ge 0$

Moreover, $A\bar{x}\le 0$

(P) is bounded, so $c^T\bar{x}\le 0$

So $-c^T\bar{x}+b^T\bar{y}\ge 0$ — contradicting (2) ■

Proof of Strong Duality Theorem for Standard Inequality Form

Note that $\bar{x}$ is optimal for

$$(\tilde{P})\begin{cases}\max c^Tx\\ \text{subject to } \begin{bmatrix}A\\ -I\end{bmatrix}x\le\begin{bmatrix}b\\ 0\end{bmatrix}\end{cases}$$

The dual of $(\tilde{P})$ is

$$(\tilde{D})\begin{cases}\min & b^Ty\end{cases}$$

Standard Equality Form

$(P3)$ is in standard equality form

$$(PSE)\begin{cases} \max c^Tx \\ \text{subject to} \\ Ax = b \\ x \geq 0 \end{cases}$$

The dual of (PSE) is

$$(DSE)\begin{cases} \min b^Ty \\ \text{subject to } A^Ty \geq c \end{cases}$$

Theorem (Strong Duality for Standard Inequality Form)

If (PSI) has an optimal solution $\bar{x}$ then (DSI) has an optimal solution $\bar{y}$ and $c^T\bar{x} = b^T\bar{y}$.

Corollary

If (DSI) has an optimal solution $\bar{y}$ then (PSI) has an optimal solution $\bar{x}$, and $c^T\bar{x} = b^T\bar{y}$.
(That is, "the dual of (DSI) is (PSI)")

Theorem (Strong Duality for Standard Equality Form)

If (PSE) has an optimal solution $\bar{x}$, then (DSE) has an optimal solution $\bar{y}$ and $c^T\bar{x} = b^T\bar{y}$.

Proof

Exercise

Constructing Duals

(P) max	(D) min
$\leq$ constraint	non-neg. var.
$\geq$ constraint	non-pos. var
$=$ constraint	free variable
non-neg. var.	$\geq$ constraint
non-pos. var.	$\leq$ constraint
free variable	$=$ constraint

The dual of $(\tilde{P})$ is

$$(\tilde{D})\begin{cases} \min & b^Ty \\ \text{subject to} & A^Ty - s = c \\ & y, s \geq 0 \end{cases}$$

By the Strong Duality Theorem, $(\tilde{D})$ has an optimal solution $(\bar{y}, \bar{s})$ and $c^T\bar{x} = b^T\bar{y}$. Note that since $\bar{s} \geq 0$, $\bar{y}$ is feasible for (DSI). However, $c^T\bar{x} = b^T\bar{y}$, so $\bar{y}$ is optimal for (DSI)

Proof of Corollary

Note that $\bar{y}$ is optimal for

$$(P)\begin{cases} \max & -b^Ty \\ \text{subject to} & -A^Ty \leq -c \\ & y \geq 0 \end{cases}$$

which is in standard inequality form.

The dual of (P) is

$$(D)\begin{cases} \min & -c^Tx \\ \text{subject to} & -Ax \geq -b \\ & x \geq 0 \end{cases}$$

By the Theorem, (D) has an optimal solution $\bar{x}$ and $-c^T\bar{x} = -b^T\bar{y}$

Note that $\bar{x}$ is clearly optimal for (PSI) ■

Yet Other Forms

$$(P)\begin{cases} \max & 3x_1 - x_2 + x_3 & \\ \text{subject to} & 2x_1 + 2x_2 = 4 & y_1 \\ & x_1 - 2x_2 + 2x_3 \leq 3 & y_2 \geq 0 \\ & x_1, x_2 \geq 0 & \end{cases}$$

The dual of (P) is

$$(D)\begin{cases} \min & 4y_1 + 3y_2 & \\ \text{subject to} & 2y_1 + y_2 \geq 3 & x_1 \geq 0 \\ & 2y_1 - 2y_2 \geq -1 & x_2 \geq 0 \\ & 2y_2 = 1 & x_3 \\ & y_2 \geq 0 & \end{cases}$$

Complementary Slackness

October-10-12 2:31 PM

Complementary Slackness Theorem

$(P)\ \max(c^T x:\ Ax \le b)$
$(D)\ \min(b^T y:\ A^T y = c, y \ge 0)$

Let $\bar{x}$ be feasible for (P) and $\bar{y}$ be feasible for (D). Then $c^T\bar{x} = b^T\bar{y}$ if and only if for each $i \in \{1, \dots, m\}$ either $\bar{y}_i = 0$ or $[A_{i,1}, \dots, A_{i,n}]\bar{x} = b_i$

Standard Inequality Form

Let $\bar{x}$ be feasible for
$(PSI)\ \max(c^T x : Ax \le b, x \ge 0)$
and $\bar{y}$ be feasible for
$(DSI)\ \min(b^T y : A^T y \ge c, y \ge 0)$
Then $c^T\bar{x} = b^T\bar{y}$ iff

1) For each $i \in \{1, \dots, m\}$ $[A_{i,1}, \dots, A_{i,n}]\bar{x} = b_i$ or $\bar{y}_i = 0$; and
2) For each $j \in \{1, \dots, n\}$ $[A_{1,j}, \dots, A_{m,j}]\bar{y} = c_j$ or $\bar{x}_j = 0$

Proof: Exercise

Standard Equality Form

Let $\bar{x}$ be feasible for
$(PSE)\ \max(c^T x : Ax = b, x \ge 0)$
and $\bar{y}$ be feasible for
$(DSE)\ \min(b^T y : A^T y \ge c)$
Then $b^T\bar{y} = c^T\bar{x}$ if and only if for each $j \in \{1, \dots, n\}$, either $[A_{1,j}, \dots, A_{n,j}]\bar{y} = c_j$ or $\bar{x}_j = 0$

Proof of Complementary Slackness Theorem

Consider
$(P)\ \max(c^T x:\ Ax \le b)$
and its dual
$(D)\ \min(b^T y:\ A^T y = c, y \ge 0)$

If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D) then

$$b^T\bar{y} - c^T\bar{x} = \bar{y}^T b - \bar{y}^T A\bar{x} = \bar{y}^T(b - A\bar{x}) = \sum_{i=1}^{m} \bar{y}_i\left(b_i - \sum_{j=1}^{n} A_{ij}\bar{x}_j\right)$$

$$\bar{y}_i \ge 0, \left(b_i - \sum_{j=1}^{n} A_{ij}\bar{x}_j\right) \ge 0 \text{ so } \bar{y}_i\left(b_i - \sum_{j=1}^{n} A_{ij}\bar{x}_j\right) \ge 0$$

Equality holds if and only if either $\bar{y}_i = 0$ or $\sum_{j=1}^{n} A_{ij}\bar{x}_j = b_i$

Proof of Standard Inequality form CST

If $\bar{x}$ is feasible for (PSI) and $\bar{y}$ is feasible for (DSI) then
$b^T\bar{y} - c^T\bar{x} \ge \bar{y}^T b - \bar{y}^T A\bar{x} = \bar{y}^T(b - A\bar{x})$
$b^T\bar{y} - c^T\bar{x} \ge \bar{x}^T A^T\bar{y} - \bar{x}^T c = x^T(A^T\bar{y} - c)$
$\bar{y} \ge 0, \quad b - A\bar{x} \ge 0 \Rightarrow \bar{y}^T(b - A\bar{x}) \ge 0$
$\bar{x} \ge 0, \quad A^T\bar{y} - c \ge 0 \Rightarrow x^T(A^T\bar{y} - c) \ge 0$

If $b^T\bar{y} - c^T\bar{x} = 0$ then
$\forall i \in \{1, \dots, m\}\ \bar{y}_i = 0$ or $(b_i - (A\bar{x})_i) = 0$
$\forall j \in \{1, \dots, n\}\ \bar{x}_j = 0$ or $\left((A^T\bar{y})_i - c_j\right) = 0$

Conversely,
$\bar{y}^T(b - A\bar{x}) = 0 = -\bar{x}^T(A^T\bar{y} - c)$
$b^T\bar{y} - \bar{y}^T A\bar{x} = -\bar{y}^T A\bar{x} + c^T\bar{x}$
$b^T\bar{y} - c^T\bar{x} = 0$
■

Proof of Standard Equality Form CST

Rewrite (DSE) as
$(DSE')\ \max(-b^T y : -A^T y \le -c)$
and apply the original complementary slackness theorem. ■

Simplex Method

October-10-12 2:54 PM

Consider

$$(P)\begin{cases}\text{maximize} & c^T x\\ \text{subject to} & Ax = b\\ & x \ge 0\end{cases}$$

and its dual

$$(D)\begin{cases}\text{minimize} & b^T y\\ \text{subject to} & A^T y \ge c\end{cases}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$

We can assume that $rank(A) = m$ (without loss of generality)

Basic Solution

$A = [A_1, \dots, A_n]$ and for $B \subseteq \{1, \dots, n\}$, $A_B = [A_i : i \in B]$

We call B a **basis** if $|B| = m$ and $rank(A_B) = m$

For a basis B,

1) There is a unique solution to
$$\begin{cases} Ax = b\\ x_j = 0, j \notin B\end{cases}$$
This is a **basic solution** for B
2) There is a unique $y \in \mathbb{R}^m$ satisfying
$(A_B)^T y = c_B$
this is the **basic dual solution.**

If $\bar{x}$ is the basic solution for B and $\bar{x} \ge 0$, then we call $\bar{x}$ a **basic feasible solution**.

If $\bar{y}$ is the basic dual solution for B and $A^T\bar{y} \ge c$, then we call $\bar{y}$ a **basic dual feasible solution**.

Optimality Theorem

Let $\bar{x} \in \mathbb{R}^n$ be the basic solution for B and $\bar{y} \in \mathbb{R}^m$ be the basic dual solution for B. Then $c^T\bar{x} = b^T\bar{y}$. Moreover, if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible, then $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D).

Remarks

1) $\bar{x} \in \mathbb{R}^n$ is an extreme point of $\{x \in \mathbb{R}^n : Ax = b,\ x \ge 0\}$ iff it is a basic feasible solution. (See assignment 2)
2) $\bar{y} \in \mathbb{R}^m$ is an extreme point of $\{y \in \mathbb{R}^m : A^T y \ge c\}$ iff it is a basic dual feasible solution. (See assignment 1)

Claim

A feasible solution for (P) is a basic feasible solution iff the columns of $[A_j : \bar{x}_j \neq 0]$ are linearly independent.

Proof

$\Rightarrow$ By definition

$\Leftarrow$ any linearly independent set extends to a basis ■

Simplex Method

Using the linear program definitions given at the top of the page.

Let $\bar{x}$ be a basic feasible solution for a basis B, let $\bar{y}$ be the basic dual solution for B, and let $\bar{v} = c^T\bar{x}\ (= b^T\bar{y})$

Recall $(A_B)^T\bar{y} = c_B$

Note that, for any feasible x,

$c^Tx = c^Tx - \bar{y}^T(Ax - b) = (c - A^T\bar{y})^T x + \bar{y}^T b = (c - A^T\bar{y})^T x + \bar{v}$

We can rewrite (P) as

$$(P')\begin{cases}\text{maximize} & \bar{c}^T x + \bar{v}\\ \text{subject to} & \bar{A}x = \bar{b}\\ & x \ge 0\end{cases}$$

Where

1) $\bar{c} = c - A^T\bar{y}$
2) $\bar{A} = (A_B)^{-1}A$
3) $\bar{b} = (A_B)^{-1}b$

Note that

i) $\bar{A}_B = I$ so we may assume that the rows of $\bar{A}$ are indexed by the elements of B and that $\bar{b}$ is indexed by B.
ii) $\bar{x}_B = \bar{b}$
iii) $\bar{c}_B = c_B - A_B^T\bar{y} = 0$ (by the selection of $\bar{y}$)
iv) $\bar{y}$ is feasible for (D) iff $\bar{c} \le 0$

Optimality

If $\bar{c} \le 0$, then $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D).

Proof of Optimality Theorem

$b^T\bar{y} - c^T\bar{x} = \bar{x}^TA^T\bar{y} - \bar{x}^Tc = \bar{x}^T(A^T\bar{y} - c) = \bar{x}_B^T(A_B^T\bar{y} - c_B)$ ■

Note: This proof works since $\bar{x}$ and $\bar{y}$ satisfy the complementary slackness conditions:

If $i \in B$ then $A_i^T y = c_i$ else $x_i = 0$

Finding a Basic Feasible Solution

Input: A feasible solution $\bar{x}$

Output: A basic feasible solution

Step 1

If $[A_j : \bar{x}_j \neq 0]$ has independent columns then STOP: output $\bar{x}$

Step 2

Find $d \in \mathbb{R}^n$ such that

1) $Ad = 0$
2) $d_j = 0$ whenever $\bar{x}_j = 0$
3) $d \neq 0$

Step 3

If $d \le 0$, replace d with $-d$.

Let $\lambda = \max(t \in \mathbb{R} : \bar{x} - td \ge 0)$

Replace $\bar{x}$ with $\bar{x} - \lambda d$

Repeat from step 1

Note that $|support(\bar{x})|$ decreases with each iteration, so the algorithm terminates, and by the claim, the solution returned is basic.

Simplex Method

Goal

Given a basic feasible solution, solve (P)

Simplex Method Example

$$(P)\begin{cases}\text{maximize} & 2x_1 + 3x_2\\ \text{subject to} & \\ & x_1 + x_3 - x_4 = 4\\ & x_2 - x_3 + 2x_4 = 2\\ & -x_1 + x_2 + x_5 = 4\\ & x_1, \dots, x_5 \ge 0\end{cases}$$

Note that $\{1,2,5\}$ is a basis

For any feasible x, $2x_1 + 3x_2 = 2(4 - x_3 + x_4) + 3(2 + x_3 - 2x_4) = 14 + x_3 - 4x_4$

(Here we are eliminating the basic variables from the objective function.)

So (P) is equivalent to

$$(P_1)\begin{cases}\text{maximize} & 14 + x_3 - 4x_4\\ \text{subject to} & \\ & x_1 + x_3 - x_4 = 4\\ & x_2 - x_3 + 2x_4 = 2\\ & 2x_3 - 3x_4 + x_5 = 6\\ & x_1, \dots, x_5 \ge 0\end{cases}$$

Note that the linear systems of (P) and (P_1) are equivalent

Warning: (P) and (P_1) have different duals

The basic solution is

$\bar{x} = [4, 2, 0, 0, 6]^T$

and the objective value is 14

Note that x_3 has a positive coefficient in the objective function for (P_1)

Set $\tilde{x}_3 = t$ and $\tilde{x}_4 = 0$. Now solve for $\tilde{x}_1, \tilde{x}_2, \tilde{x}_5$

$$\tilde{x} = \begin{bmatrix}4\\2\\0\\0\\6\end{bmatrix} - t\begin{bmatrix}1\\-1\\-1\\0\\2\end{bmatrix}$$

Take $t = 3$ we get

$$\tilde{x} = \begin{bmatrix}1\\5\\3\\0\\0\end{bmatrix}$$ with objective value $= 17$

This is basic for $B = \{1,2,3\}$

Eliminate the new basic variables from the objective function

$14 + x_3 - 4x_4 = 14 + \frac{1}{2}(6 + 3x_4 - x_5) - 4x_4 = 17 - \frac{5}{2}x_4 - \frac{1}{2}x_5$

Optimality

If $\bar{c} \leq 0$, then $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D).

Iteration

Suppose that $\bar{c}_j > 0$ for some j. (Note that $j \notin B$ by ii))
x_j is the **entering variable**.

Define $\bar{d} \in \mathbb{R}^n$ by

$$\bar{d}_i = \begin{cases} -\bar{a}_{ij} & : \quad i \in B \\ 1 & : \quad i = j \\ 0 & : \quad \text{otherwise} \end{cases}$$

Note that the unique solution to

$$\begin{cases} \bar{A}x = \bar{b} \\ x_j = t \\ x_i = 0, i \notin B \cup \{j\} \end{cases}$$

is $\bar{x} + td$, which has objective value $\bar{v} + t\bar{c}_j$ (in (P))

Unboundedness

If $\bar{d} \geq 0$, (P) is unbounded.
$\{\bar{x} + t\bar{d} : t \geq 0\}$ is a feasible halfline and $\bar{c}^T\bar{d} = \bar{c}_j > 0$

Update

Suppose that $\bar{d}$ has a negative entry (otherwise unbounded).
Choose $t = max(\lambda \in \mathbb{R} : \bar{x} + \lambda\bar{d} \geq 0)$ and replace $\bar{x}$ with $\bar{x} + t\bar{d}$
By our choice of t, there exists $i \in B$ such that $\bar{x}_i = 0$ and $\bar{d}_i < 0$
$\bar{x}_i$ is the **leaving variable**.

$B - \{i\} + \{j\}$ is a basis.
Replace B with $B - \{i\} + \{j\}$
Note that $\bar{x}$ is a basic solution for B.
Now we repeat.

Since the basis has changed in only two elements, it is easy to update the problem (P')

Eliminate the new basic variables from the objective function

$$14 + x_3 - 4x_4 = 14 + \frac{1}{2}(6 + 3x_4 - x_5) - 4x_4 = 17 - \frac{5}{2}x_4 - \frac{1}{2}x_5$$

For any non-negative x, we get an objective value ≤ 17

Therefore, $\tilde{x} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ is optimal.

Termination of the Simplex Method

October-17-12 3:12 PM

Termination

- There are $\leq \binom{n}{m}$ bases
- At each iteration the objective value does not decrease
- There are examples where the Simplex Method cycles
    - That is, it revisits a basis
    - The easiest way to avoid cycles is to pick entering variables randomly amongst the choices.
- If the objective values does not increase in an iteration, then the solution $\bar{x}$ is basic for two distinct bases B_1 and B_2. Hence $|support(\bar{x})| < m$.

Nondegeneracy

A basic solution $\tilde{x}$ is nondegenerate if $|support(\tilde{x})| = m$
(P) is nondegenerate if each of its basic solutions is nondegenerate.

Note

The Simplex Method will terminate given any nondegenerate linear program and the number of iterations is at most $\binom{n}{m}$

Hirsch Conjecture (1957)

The distance between any two vertices in 1-skeleton of (P) is $\leq m$
This is the graph-theory distance (how many edges you need to traverse.)

False (Proved in 2010)

The 1-skeleton is the frame of lines between vertices that could be traversed by the Simplex Algorithm.

Problems

1) Is there a polynomial bound on the diameter of the 1-skeleton?
2) Is there a "pivoting rule" for the simplex Method that gives a polynomial-time algorithm?

    Pivoting rule is the rule for selecting an entering variable of possible choices.

Perturbation Method

Idea: We carefully select the leaving variable in order to avoid cycling. This is achieved by perturbing b.

$$(P)\begin{cases} \text{max} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{cases}$$

Consider

$$(P')\begin{cases} \text{max} & c^T s \\ \text{subject to} & Ax = b' \\ & x \geq 0 \end{cases}$$

where $b' = \begin{bmatrix} b_1 + \epsilon \\ b_2 + \epsilon^2 \\ \vdots \\ b_n + \epsilon^n \end{bmatrix}$

here ϵ is a variable that we think of a s a small positive real number.

Polynomial Ordering

For polynomials $p(\epsilon)$ and $q(\epsilon)$ we write $p(\epsilon) < q(\epsilon)$ (for arbitrarily small ε) if the coefficient of the smallest degree term of $q(\epsilon) - p(\epsilon)$ is positive.

Claim

(P') is nondegenerate

Note that we can solve (P) using the Simplex Method since it is nondegenerate.

Proof of Claim

For a basis B consider the basic solution $\bar{x}$. We have $\bar{x}_B = (A_B)^{-1}b'$
Since each row of $(A_B)^{-1}$ is a nonzero real vector and the entries of b are polynomials with distinct degrees, each term of $\bar{x}_B$ is nonzero. ■

Another Way to Avoid Cycling

Break ties when choosing entering and leaving variables by taking the one of minimum subscript.

Theorem (Bland)

The smallest subscript rule avoids cycling.

Feasibility

October-19-12 2:35 PM

Information Returned by SM

If has optimal solution, it will give you the solution and the feasible dual solution.

If unbounded if will find an unbounded ray.

The dual basic solution to the optimal value of (P') is a proof that (P) is infeasible if that is the case.

Finding a Feasible Solution

Consider

$$(P)\begin{cases}\text{maximize} & c^T x\\ \text{subject to} & Ax = b\\ & x \geq 0\end{cases}$$

We have algorithms for

1) Given a feasible solution, find a basic feasible solution.
2) Given a basic feasible solution, solve (P)

How do you find a feasible solution?
We can scale so that $b \geq 0$.
Consider the following **auxiliary problem**

$$(P')\begin{cases}\text{maximize} & -s_1 - s_2 \dots - s_m\\ \text{subject to} & Ax + s = b\\ & x, s \geq 0\end{cases}$$

Note that

1) $x = 0,\ s = b$ is a (basic) feasible solution to (P'), so we can solve (P') using the Simplex Method.
2) Since $-s_1 - s_2 - \cdots - s_m \leq 0$, (P') is bounded so the simplex method will terminate with an optimal solution $(\bar{x}, \bar{s})$
3) If $\bar{s} = 0$ then $\bar{x}$ is a feasible solution to (P)
4) If $\tilde{x}$ is feasible for (P) then $(\tilde{x}, 0)$ is an optimal solution for (P')

Hence the optimal value for (P') is zero iff (P) has a feasible solution.

Remark

If $(\bar{x}, 0)$ is a (basic?) feasible solution for (P') then $\bar{x}$ is a basic feasible solution for (P).

Information Returned by SM

If has optimal solution, it will give you the solution and the feasible dual solution.

If unbounded if will find an unbounded ray.

Recall: Farkas' Lemma

Exactly one of the following has a solution

1) $Ax = b, x \geq 0$
2) $A^T y \geq 0, \quad b^T y < 0$

The dual of (P')

$$(D')\begin{cases}\text{min} & b^T y\\ \text{subject to} & A^T y \geq 0\\ & y \geq -1\end{cases}$$

If (P) is infeasible and $\bar{y}$ is an optimum solution to (D') then $b^T\bar{y} < 0$ so $\bar{y}$ satisfies $A^T y \geq 0,\ b^T y < 0$

Note: this gives a (more) constructive proof of the Farkas' Lemma

The dual basic solution to the optimal value of (P') is a proof that (P) is infeasible if that is the case.

Midterm Review

October-22-12 2:33 PM

Geometry

Convex Hull and Cone

For $z^1, \dots, z^n \in \mathbb{R}^m$, define
$conv(z^1, \dots, z^n) = \{\lambda_1 z^1 + \cdots + \lambda_n z^n : \lambda \in \mathbb{R}, \quad \lambda \geq 0, \quad \lambda_1 + \cdots \lambda_n = 1\}$
$cone(z^1, \dots, z^n) = \{\lambda_1 z^1 + \cdots + \lambda_n z^n : \lambda \in \mathbb{R}, \quad \lambda \geq 0\}$

Separating Hyperplane Theorem (Farkas Lemma)

1) If $b \neq conv(z^1, \dots, z^n)$ then there is a hyperplane separating b from the convex hull $conv(z^1, \dots, z^n)$
2) If $b \neq cone(z^1, \dots, z^n)$ then there is a hyperplane separating b from the convex hull $cone(z^1, \dots, z^n)$

Polyhedral Theory

Polyhedron: $\{x \in \mathbb{R}^n : Ax \leq b\}$
Polytope: bounded polyhedron
Polyhedral cone: $\{x \in \mathbb{R}^n : Ax \leq 0\}$

Lemma 1

For a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, the following are equivalent:
1) P has no extreme point
2) P contains a line
3) $rank(A) < n$

Lemma 2

Characterization of extreme points:
For $Ax \leq b$, let $A'x \leq b'$ be the subsystem of $Ax \leq b$ satisfied by x with equality.
Then x is an extreme point iff $rank(A') = n$
⇒ There are only finitely many extreme points

Theorem A

$S \subseteq \mathbb{R}^n$ is a polytope iff it is the convex hull of a finite set of points in $\mathbb{R}^n$

Theorem B

If S is a polyhedral cone, then there is a finite set $Z \subseteq \mathbb{R}^n$ such that $s = cone(Z)$

Remark

☐ The converse is true (Exercise; use Theorem A)

For $S_1, S_2 \subseteq \mathbb{R}^n$, define
$S_1 + S_2 = \{a + b : a \in S_1, b \in S_2\}$

Theorem C

Let Z be the set of extreme points of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.
If P does not contain a line, then $P = conv(Z) + \{x \in \mathbb{R}^n : Ax \leq 0\}$

Remark

B & C ⇒ There exist finite $Z, D \subseteq \mathbb{R}^n$ such that
1) $P = conv(Z) + cone(D)$
   We used that P does not contain a line, but can easily account for lines by having $d, -d \in D$ where $\{x + \lambda d\}$ is a line.
   Note we can scale that
2) $\|d\| = 1$ for each $d \in D$
   If (P) does not contain a line then there are unique minimal sets $Z, D \subseteq \mathbb{R}^n$ satisfying (1) and (2).
   Z is the set of extreme points.
   D is the set of **extreme rays**. (Take a plane through the cone and look at the extreme points)
   ⇒ Every polyhedron that does not contain a line is generated by its extreme points and extreme rays.

Applications

Carathéodory's Theorem
Helly's Theorem

Linear Programming

$$(P) \begin{cases} \quad \max & c^T x \\ \text{subject to} & Ax \leq b \end{cases}$$

$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

Fundamental Theorem

(P) is either infeasible, unbounded or has an optimal solution.

Infeasibility Theorem (Farkas Lemma)

(P) is infeasible iff there exists $y \in \mathbb{R}^m$ satisfying $(A^T y = 0,\ b^T y < 0, y \geq 0)$

Unboundedness Theorem

(P) is unbounded iff
- (P) is feasible, and
- there exists $d \in \mathbb{R}^n$ satisfying $(Ad \leq 0,\ c^T d > 0)$.

Dual

The dual of (P) is

$$(D)\begin{cases} \min & b^T y \\ \text{subject to} & A^T y = c \\ & y \geq 0 \end{cases}$$

Weak Duality Theorem

If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D) then $c^T\bar{x} \leq b^T\bar{y}$.

Strong Duality Theorem

(P) has an optimal solution iff (D) has an optimal solution.
Moreover, if $\bar{x}$ is optimal for (P) and $\bar{y}$ is optimal for (D), then
$c^T\bar{x} = b^T\bar{y}$

Application of Duality

Theorem

If $\bar{x}$ is an extreme point of the polyhedron
$P = \{x \in \mathbb{R}^n : Ax \leq b\}$, then there is a halfspace H such that $P \cap H = \{\bar{x}\}$

Exercise

- [ ] Let $\bar{x}$ be an extreme point of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$,
where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$.
Show that, if $\bar{z} \notin \mathbb{Z}^n$ then there exists $c \in \mathbb{Z}^n$ such that $\bar{x}$ s an optimal solution to $\max(c^T x : x \in P)$ and $c^T\bar{x} \notin \mathbb{Z}$.

Do you need A, b to be integer valued?

Proof of Strong Duality Theorem

Ideally we would like $\bar{x}, \bar{y}$ with $c^T\bar{x} = b^T\bar{y}$.
That is, we want $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ satisfying

$$(1)\begin{cases} -c^T x + & b^T y & = 0 & \bar{z} \\ Ax & & \leq b & \bar{y} \geq 0 \\ & -A^T y & = -c & \bar{x} \\ & y & \geq 0 & \end{cases}$$

Suppose no such x, y exist.

By the Farkas Lemma, there exist $z \in F, x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$ satisfying

$$(2)\begin{cases} -c^T x + & b^T x & < 0 \\ Ax & & \leq bz \\ & A^T y & = cz \\ & \bar{y} & \geq 0 \\ & \bar{z} & \geq 0 \end{cases}$$

Claim: $z = 0$
Proof: Otherwise we can scale to get $z = 1$ and then (x, y) satisfy (1), contradiction ■

Either:
1) x satisfies $(c^T x > 0, Ax \leq 0)$ or
2) y satisfies $(b^T y < 0, A^T y = 0, y \geq 0)$

In case 1, (P) is infeasible or unbounded and (D) infeasible
In case 2, (D) is infeasible or unbounded and (P) is infeasible
In either case, neither (P) nor (D) has an optimal solution.

Proof of Polyhedron-Point-Halfspace Theorem

Since $\bar{x}$ is an extreme point, there exists a partition $A'x \leq b'$, $A''x \leq b''$ of the inequalities $Ax \leq b$ such that $A'\bar{x} = b'$, $rank(A') = n$ and A' is $n \times n$. ($\bar{x}$ may satisfy some of $A''x \leq b''$ with equality.)

Let
$c = (A')^T \underline{1}$
$\alpha = c^T\bar{x} = \underline{1}^T A'x = \underline{1}^T b'$
$H = \{x \in \mathbb{R}^n : c^T x \geq \alpha\}$

Now consider the LP:

$$(P)\begin{cases} \max & c^T x \\ \text{subject to} & A'x \leq b' \\ & A''x \leq b'' \end{cases}$$

and its dual

$$(D)\begin{cases} \min & (b')^T y + (b'')^T z \\ \text{subject to} & (A')^T y + (A'')^T z = c \\ & y, z \geq 0 \end{cases}$$

Let $\bar{y} = 1$ and $\bar{z} = 0$
Now $\bar{x}$ is feasible for (P), and $\bar{y}, \bar{z}$ is feasible for (D) with
$c^T\bar{x} = (b')^T\bar{y} + (b'')^T\bar{z} = \alpha$

So $\bar{x}$ is optimal for (P) and $(\bar{y}, \bar{z})$ is optimal for (D).

Consider another optimal solution $\hat{x}$ for (P). Note that $\bar{y} > 0$, so by the complementary slackness conditions, $A'\hat{x} = b'$. However A' is invertible, so $\hat{x} = \bar{x}$. Hence $\bar{x}$ is the unique optimal solution and $H \cap P = \{\bar{x}\}$. ■

Integer Programming

October-26-12 2:33 PM

Integer Program

An integer program is a problem of the form

$$(IP)\begin{cases}\max & c^T x\\ \text{subject to} & Ax \le b\\ & x \in \mathbb{Z}^n\end{cases}$$

Linear Programming Relaxation

Drop the integer constraint.
The linear programming relaxation is

$$(LP)\begin{cases}\max & c^T x\\ \text{subject to} & Ax \le b\end{cases}$$

Notation

We denote by $OPT(IP)$ and $OPT(LP)$ to be optimal values of (IP) and (LP) respectively.

For infeasible problems we define $OPT = -\infty$ and for unbounded problems we define $OPT = \infty$.

Note that the $OPT(IP) \le OPT(LP)$ since the optimal solution for (IP) is feasible for (LP)

If z is the set of feasible solutions to (IP) then,

$$conv(z) \subseteq \{x \in \mathbb{R}^n : Ax \le b\}$$

equality is "rare".

Integral

A polyhedron is **integral** if its extreme points are integral. (That is, all entries are integers).

Lemma (1)

If $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is integral, $rank(A) = n$, and (LP) has an optimal solution, then $OPT(IP) = OPT(LP)$

Totally Unimodular Matrices

A matrix is totally unimodular (TU) iff each of its square submatrices has determinant 0, 1, or -1.
In particular, the entries are 0, 1, -1

Theorem

Let $A \in \{0, \pm 1\}^{m \times n}$ be TU and $b \in \mathbb{Z}^m$ then $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is integral.

Modifying TU Matrices

Let $A \in \{0, \pm 1\}^{m \times n}$ be TU. then
(1) A^T is TU
(2) $[I, A]$
(3) If A' is obtained from A by scaling a row or column by -1, then A' is TU
(4) $[A, -A]$ is TU

$\Rightarrow$ For $b \in \mathbb{Z}^m$, the following polyhedra are integral

- $P_1 = \{x \in \mathbb{R}^n : Ax \le b, \quad x \ge 0\}$
- $P_2 = \{x \in \mathbb{R}^n : Ax = b, \quad x \ge 0\}$
- $P_3 = \{x \in \mathbb{R}^n : Ax \le b, \quad l \le x \le u$ where $l, u \in \mathbb{Z}^n\}$

Lemma (2)

Let $A \in \{0, \pm 1\}^{m \times n}$. If each column of A contains at most 1 and at most one -1 then A is a TU.

Proof of Lemma (1)

$OPT(LP)$ is attained at an extreme point.

Proof of Theorem

Let $\bar{x} \in \mathbb{R}^n$ be an extreme point of P. Then by Assignment 1, there is a subsystem $A'x \le b'$ of $Ax \le b$ such that $A'\bar{x} = b'$
A' is $n \times n$ and $rank(A') = n$

Now $\bar{x} = (A')^{-1}b'$. By Crammer's rule, $(A')^{-1}$ is integral and hence so is $\bar{x}$.

Proof that P_2 is Integral

P_1 can be rewritten as

$$\left\{x \in \mathbb{R}^n : \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}\right\}$$

By applying (1), (2), and (4) $\begin{bmatrix} A \\ -A \\ -I \end{bmatrix}$ is unimodular
so by the theorem, P_2 is integral.

Proof of Lemma (2)

Suppose otherwise and consider a counterexample $A \in \{0, \pm 1\}^{m \times n}$ with $m + n$ as small as possible.
Thus $m = n$ and $\det(A) \ne \{0, 1, -1\}$
By minimality, each column contains has two nonzero entries (otherwise expanding the determinant at that column would give a smaller matrix with the same property.
The two nonzero entries have to be 1 and -1, but then the rows sum to zero and hence, $\det(A) = 0$. Contradiction ■

Graph Theory

October-29-12 2:32 PM

Incidence Matrix
Vertices on rows, edges on columns. 1 if that edge is incident on that vertex, 0 otherwise.

Bipartite Graph
A graph $G = (V, E)$ is bipartite with bipartition (X, Y) if (X, Y) is a partition of V and each edge has an end in X and an end in Y.

Theorem
The incidence matrix of a bipartite graph is TU.

Remark
Suppose that $x \in \{0,1\}^E$ and $Ax = b$. (A is the incidence matrix of G and $b \in \mathbb{Z}^V$)
Then b_v is the number of edges in support(x) that vertex v is incident with.

Matching
$M \subseteq E$ is a **matching** of G if each vertex is incident to at most one edge in M
A **perfect matching** is a matching that **saturates** all vertices. So if $\widetilde{M}$ is a perfect matching then $|\widetilde{M}| = \frac{1}{2}|V|$

Matching Polyhedra
Let A be the incidence matrix of G. Define
$M(G) = \{x \in \mathbb{R}^E : Ax \leq \mathbf{1}, \quad x \geq 0\}$
$PM(G) = \{x \in \mathbb{R}^E : Ax = \mathbf{1}, \quad x \geq 0\}$

Note that
1) $M(g) \cap \mathbb{Z}^E \subseteq \{0,1\}^E$ and $PM(G) \cap \mathbb{Z}^E \subseteq \{0,1\}^E$
2) For $x \in \{0,1\}^E$, $x \in M(G)$ iff support(x) is a matching.
3) For $x \in \{0,1\}^E$, $x \in PM(G)$ iff support(x) is a perfect matching.
4) For bipartite G, A is TU so $M(G)$ and $PM(G)$ are integral polyhedra.

Let $M(G)$ be the set of all $x \in \{0,1\}^E$ such that support(x) is a matching.

Theorem (Summary)
If G is bipartite, then
$conv(M(G)) = \{x \in \mathbb{R}^E : Ax \leq \mathbf{1}, x \geq 0\}$

r-Regular
A graph is r-regular if each of its vertices has degree r.

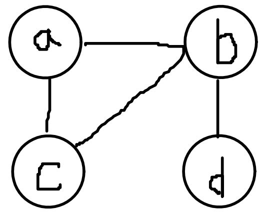
Theorem
For $r \geq 1$, every r-regular bipartite graph has a perfect matching.

Cover
$C \subseteq V$ is a **cover** of G if each edge of G is incident with a vertex in C.

Note that if M is a matching and C then $|M| \leq |C|$.

König's Theorem
In a bipartite graph, the maximum size of a matching is equal to the minimum size of a cover.

Graph Example



Graph $G(V, E)$
Finite vertex set: $V = \{a, b, c, d\}$
$E = \{ab, ac, cb, bd\}$ is a set of unordered pairs of vertices, called **edges**.

Incidence Matrix
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} ab & ac & bc & bd \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$
The row sum for the v^{th} row of A is the number of edges of G incident with the vertex v and is denoted deg(v).

Note that A is not TU:
$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \pm 2$$

Proof of Theorem
Let (X, Y) be a bipartition of G and let A be the incidence matrix of G. Let A' be obtained from A by scaling the rows indexed by X by -1. By Lemma (2), A' is TU and hence so is A ■

Proof of Theorem (r-regular perfect matching)
Let G be an r-regular bipartite graph and let $x = \left[\frac{1}{r}, \dots, \frac{1}{r}\right]^T$.
Note that $x \in PM(G)$. Let $\bar{x}$ be an extreme point. Since G is bipartite, $PM(G)$ is integral, so $\bar{x} \in \mathbb{Z}^E$. Hence support$(\bar{x})$ is a perfect matching. ■

Proof of König's Theorem
Let A be the incidence matrix of a bipartite graph $G(V, E)$.
Consider
$$(P) \begin{cases} \max & \mathbf{1}^T x \\ \text{subject to} & Ax \leq \mathbf{1} \\ & x \geq 0 \end{cases}$$
The dual is
$$(D) \begin{cases} \min & \mathbf{1}^T y \\ \text{subject to} & A^T y \geq \mathbf{1} \\ & y \geq 0 \end{cases}$$
Both are feasible, with $x = 0$ and $y = 1$
Hence there exist optimal solutions for (P) and (D).
Let $\bar{x}$ be an optimal extreme point for (P) and $\bar{y}$ be an optimal extreme point for (D).
Since A is totally unimodular, $\bar{x}$ and $\bar{y}$ are integral.

Note that $\bar{x}$ and $\bar{y}$ are $\{0,1\}$-valued.
Let $\bar{M} =$ support$(\bar{x})$ and $\bar{C} =$ support$(\bar{y})$.
$\bar{M}$ is a matching and $\bar{C}$ is a cover and $|\bar{C}| = \mathbf{1}^T \bar{y} = \mathbf{1}^T \bar{x} = |\bar{M}|$ (by strong duality)
Therefore, $\bar{M}$ is a maximum matching and $\bar{C}$ is minimum cover, and they have the same size. ■

Min-Cost Matching

October-31-12 2:51 PM

Minimum-Cost Perfect Matching

Problem

Given a bipartite graph $G = (V, E)$ and $c \in \mathbb{R}^E$, find a perfect matching M minimizing $\sum_{e \in M} c_e$

We denote $\sum_{e \in M} c_e$ by $c(M)$

We will assume that G has a perfect matching.
(Either find one with an un-weighted algorithm or throw in very high weight edges)

Min-Cost Matching

Let A be the incidence matrix of G and consider

$$(P)\begin{cases} \min & c^T x \\ \text{subject to} & Ax = \mathbf{1} \\ & x \geq 0 \end{cases}$$

and its dual

$$(D)\begin{cases} \max & y(V) \\ \text{subject to} & A^T y \leq c \end{cases}$$

$$y(V) = \sum_{v \in V} y_v = \mathbf{1}^T y$$

Since A is TU, (and has at least one perfect matching), there is a perfect matching $\tilde{M}$ with $c(\tilde{M}) = OPT(P)$
Let $\bar{y} \in \mathbb{R}^V$ and $\bar{c} = c - A^T \bar{y}$ (that is $\bar{c}_e = c_e - \bar{y}_u - \bar{y}_v$ for $e = uv \in E$)
We call these **reduced costs**.
Note that $\bar{c} \geq 0$ iff $\bar{y}$ is feasible for (D)

Define $G^{=}(\bar{y})$ to be the subgraph of G with vertex set $V(G)$ and edge set $\{e \in E : \bar{c}_e = 0\}$

Complementary Slackness

If M is a perfect matching and y' is a feasible solution for (D) then $c(M) = y'(V)$ iff $M \subseteq E(G^{=}(y'))$

$E(G^{=}(y')) = \{e \in E : \bar{c}_e = 0\} = \{e \in E : c_e = A_e^T y\}$
by complementary slackness, for $e \notin E(G^{=}(y'))$, $x_e = 0$ so $M \subseteq E(G^{=}(y'))$

Claim

If $\bar{y}$ is a feasible solution for (D) and M is a perfect matching of $G^{=}(\bar{y})$, then M is a min-cost perfect matching.

Because M and $\bar{y}$ satisfy complementary slackness conditions.

Algorithm

Let (X, Y) be a partition of G. We may assume that $|X| = |Y|$. since otherwise G has no perfect matching.

Overview

0. Find a feasible $\bar{y}$ for (D)
   a. Let $\bar{y}_v = 0$ for each $v = Y$
   b. Let $\bar{y}_v = \min(c_e : e = vw, w = \text{Y})$ for $v \in \text{X}$
1. If $G^{=}(\bar{y})$ has a prefect matching M, stop. Output M
2. Find a feasible solution y' for (D) with $y'(V) > \bar{y}(V)$. Replace $\bar{y}$ with y' and goto 1.

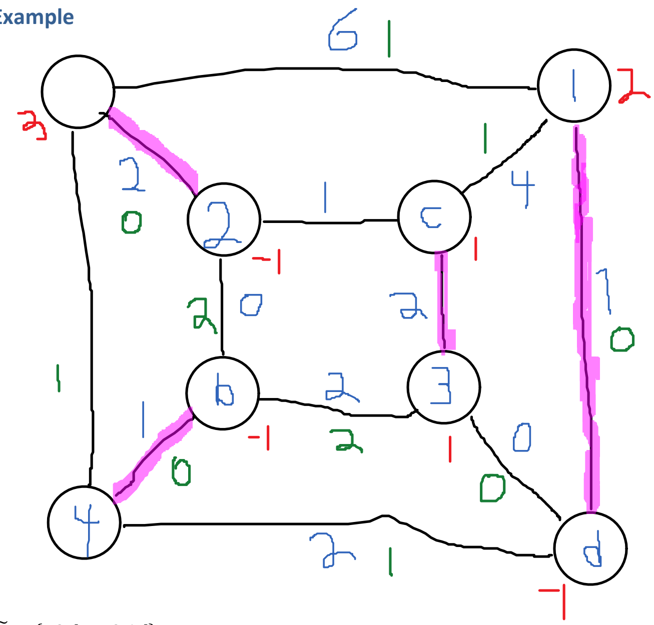
Neighbour Set

If $Z \subseteq V(G)$, the neighbour set of Z , denoted $N_G(Z)$ is the set of vertices $w \in V(G) - Z$ such that there exists $v \in Z$ such that $vw \in E(g)$

Hall's Theorem

Let G be a bipartite graph with bipartition (X, Y) where $|X| = |Y|$. Then G has a prefect matching iff for each $Z \subseteq X$, $|N(Z)| \geq |Z|$.

Example



$\tilde{M} = \{a2, b4, c3, 1d\}$

Claim

$\tilde{M}$ is a maximum matching

Suppose

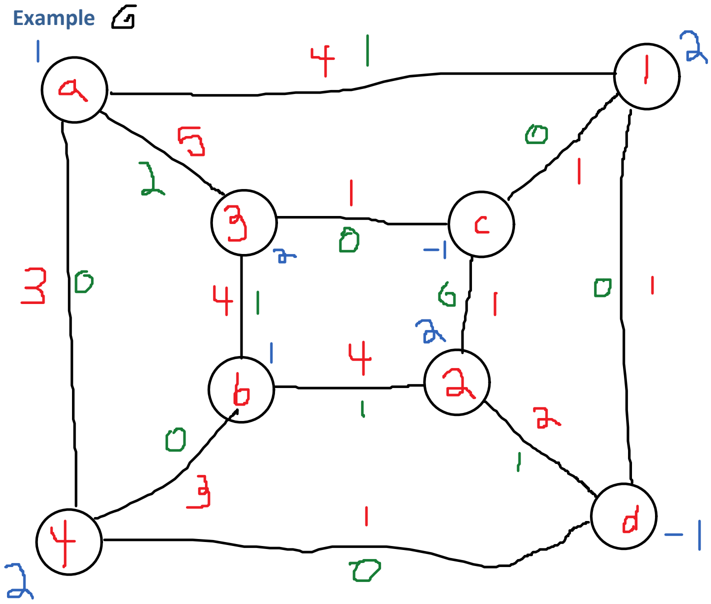
$$c'(e) = \begin{cases} c(e) + 1: & e \text{ incident with a} \\ c(e) & \text{otherwise} \end{cases}$$

Then for any perfect matching M, $c'(M) = c(M) + 1$ so finding a min cost perfect matching with respect to c' is the same as finding a min cost perfect matching with respect to c.

Define
$\bar{c}_e = c_e - \bar{y}_u - \bar{y}_v$ for each $e = uv \in E$
(In drawing, red are $\bar{y}_v$ and green are new edge weights. At this point $\bar{y}$ is specified without explanation.)

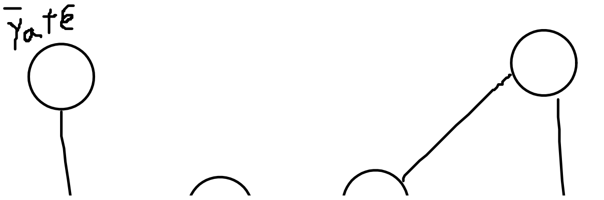
For any perfect matching M,
$\bar{c}(M) = c(M) - \bar{y}(V) = c(M) - 6$
Note that $\bar{c} \geq 0$ and $\bar{c}(\tilde{M}) = 0$, so $\tilde{M}$ is a min cost perfect matching with respect to $\bar{c}$ and hence also c.

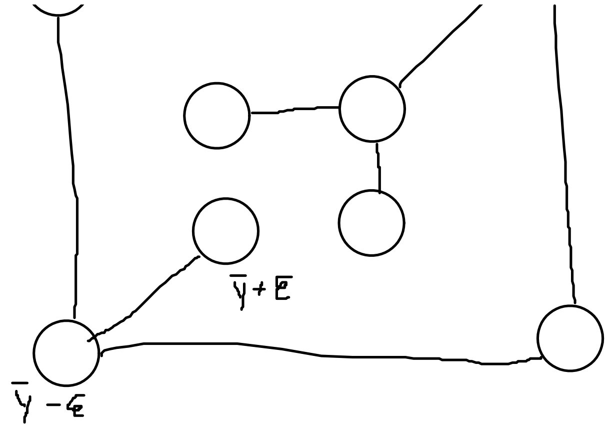
Example 6



$G^{=}(\bar{y})$

$\bar{y}_a + \varepsilon$





Note that $G^=(\bar{y})$ has no perfect matching since $N_{G^=(\bar{y})}(\{a,b\}) = \{4\}$

Change $\bar{y}$ as shown. For small ϵ, $\bar{y}$ remains feasible and we get $\bar{y}(V) = 8 + \epsilon$
We can take $\epsilon = 1$. Repeat from step 1

Proof of Hall's Theorem

$\Rightarrow$ Obvious
$\Leftarrow$
If G has no perfect matching then G has no matching of size $|X|$. So, by König's Theorem, there is a vertex cover C with $|C| < |X|$.
Let $Z = X - C$. Note that, since C is a cover, $N(Z) \leq Y \cap C$
(There are no edges between Z and $Y - C$)
Moreover, $|Z| = |X - C| > |C - X| = |Y \cap C| \geq |N(Z)|$ as required ■

Algorithm for Min-Cost Perfect Matching

November-05-12 2:33 PM

Min Cost Perfect Matching

Let $G = (V, E)$ be a bipartite graph with bipartition (X, Y) and let $c \in \mathbb{R}^E$
Assume $|X| = |Y|$

$$(P)\begin{cases} \min & c^T x \\ \text{subject to} & Ax = \mathbf{1} \\ & x \geq 0 \end{cases}$$

and its dual

$$(D)\begin{cases} \max & y(V) \\ \text{subject to} & A^T y \leq 0 \end{cases}$$

Lemma

If $\bar{y} \in \mathbb{R}^V$ is a feasible solution for (D) and $G^=(\bar{y})$ has a perfect matching M. Then M is a min cost perfect matching.

Hall's Theorem

Let G be a bipartite graph with bipartition (X, Y) where $|X| = |Y|$. Then G has a prefect matching iff $|N(Z)| \geq |Z|$ for a each $Z \subseteq X$.

Assumption

We have an efficient algorithm for Hall's Theorem.
That is, we can find either a perfect matching or a set $Z \subseteq X$ with $|N(Z)| \leq |Z|$).
See MATH 239/249

Algorithm

Let $\bar{y} \in \mathbb{R}^V$ be a feasible solution for (D) and suppose that $G^=(\bar{y})$ has no perfect matching
Then there exists $Z \subseteq \mathrm{X}$ such that $\left|N_{G^=(y)}(Z)\right| < |Z|$

Let $y'_v = \begin{cases} \bar{y}_v + \epsilon, & v \in Z \\ \bar{y}_v - \epsilon, & v \in N_{G^=(y)}(Z) \\ \bar{y}_v, & \text{otherwise} \end{cases}$

Note that, for small $\epsilon > 0$ y' is feasible and has objective value
$y'(V) = \bar{y}(V) + \epsilon\left(|Z| - \left|N_{G^=(\bar{y})}(Z)\right|\right) > \bar{y}(V)$
How large can we make ϵ?

$$\epsilon = \min\left(c_e : e = uv \in E(G), u \in Z, v \in Y - N_{G^=(y)}(Z)\right)$$

This is well defined unless $N_G(Z) = N_{G^=(y)}(Z)$
But then $|N_G(Z)| < |G|$ and hence G has no perfect matching.

Remarks

1) If $c \in \mathbb{Z}^E$ and $\bar{y} \in \mathbb{Z}^V$ then we get $y' \in \mathbb{Z}^V$.
   (So we keep an integral dual solution)
2) There is a way to chose Z so that the number of iterations is at most $|V(G)|^2$ (independent of c)
   See CO 450
3) The min cost perfect matching problem can be solved in polynomial time, even for non bipartite graphs.
   a. See CO 450.
   b. We need additional constraints for $X \subseteq V(G)$ odd. For each cut, the number of edges across the cut is ≥ 1
   c. This has exponentially many constraints, but is still solvable in polynomial time so long as the constraints are given implicitly (only generated when needed).

Directed Graphs

November-05-12 3:06 PM

Directed Graph

A directed graph is a pair (V, E) where V is a finite set and E is a set of ordered pairs of distinct vertices.
V is the **vertex set** and E is the **edge set**.
For $e = uv \in E$, u is the **tail** and v is the **head** of e.

Incidence Matrix

Like incidence matrix for undirected graph but now $A \in \{0, \pm 1\}^{V \times E}$ where

$$A_{ve} = \begin{cases} 1, & \text{if } v = \text{head}(e) \\ -1, & \text{if } v = \text{tail}(e) \\ 0, & \text{otherwise} \end{cases}$$

Since A has one 1 and one -1 in each column, A is totally unimodular.

For $X \subseteq V(G)$, we define
$\text{IN}(X) = \{uv \in E(g) : u \notin X, v \in X\}$
$\text{OUT}(X) = \{uv \in E(g) : u \in X, v \notin X\}$

Suppose that $Ax = b$
Then $x(IN(\{v\})) - x(OUT(\{v\})) = b_v$ for each $v \in V$

Notation

If $x \in \mathbb{R}^E$, $w \subseteq E$, $x(W) = \sum_{e \in W} x_e$

Inflow

$\text{inflow}_x(v) = x(\text{IN}(\{v\})) - x(\text{OUT}(\{v\}))$

(s,t)-flow

Let s and t be distinct vertices in a directed graph $G = (V, E)$ and let $u \in \mathbb{R}_+^E$ be **edge capacities**. $x \in \mathbb{R}^E$ is an (s,t) flow if $\text{inflow}_x(v) = 0$ for each $v \in V - \{s, t\}$
x is **feasible** if $0 \le x \le u$
The value of **x** is $\text{inflow}_x(v)$

Problem

Find a feasible (s,t)-flow of maximum value.
This is a linear program

$$(P) \begin{cases} \text{max} & \text{inflow}_x(t) \\ \text{subject to} & \text{inflow}_x(v) = 0, \quad v \in V - \{s,t\} \\ & 0 \le x \le u \end{cases}$$

Note that if the edge capacities are integer then this is an integer linear program.

(s,t)-cut

An (s,t)-cut is a partition (S,T) of $V(G)$ with $s \in S$ and $t \in T$, the **capacity** of (S,T) is
$u(S,T) = u(\text{IN}(T)) = u(\text{OUT}(S))$

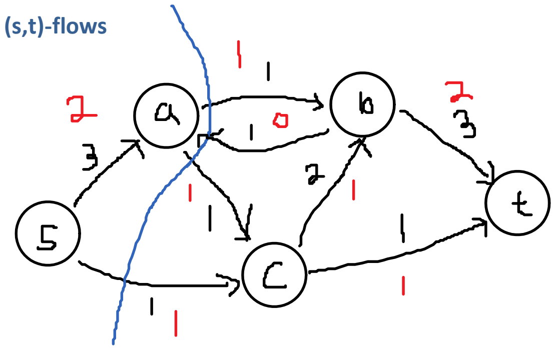
Claim

If x is a feasible (s,t)-flow and (S,T) is an (s,t)-cut, then $\text{inflow}_x(t) \le u(S,T)$

Max-Flow Min-Cut Theorem

The maximum value of a feasible (s,t)-flow is equal to the min capacity of an (s,t)-cut.

(s,t)-flows



The red edge labels give an edge flow of 3
The blue cut allows a maximum of 3 across it from s to t
Therefore 3 is optimal

Proof of Claim

$\text{inflow}_x(t) = \sum_{v \in T} \text{inflow}_x(v) = \text{inflow}_x(T) = x(\text{IN}(T)) - x(\text{OUT}(T))$
$\le u(\text{IN}(T)) = u(S,T)$
■

Proof of Max-Flow Min-Cut Theorem

We may assume that $u > 0$
Let G' be obtained from G by adding a new edge $f = ts$ and defining $u_f = \infty$ (or sum of all other capacities plus 1)
Let $c \in \mathbb{R}^E$ be given by

$$c_e = \begin{cases} 1, & e = f \\ 0, & e \ne f \end{cases}$$

Now the maximum value of a feasible (s,t)-flow is given by

$$(P) \begin{cases} \text{max} & c^T x & \\ \text{subject to} & Ax = 0 & y \in \mathbb{R}^V \\ & x \le u & z \in \mathbb{R}_+^E \\ & x \ge 0 & \end{cases}$$

The dual of (P) is

$$(D) \begin{cases} \text{min} & u^T z \\ \text{subject to} & A^T y + z \ge c \\ & z \ge 0 \end{cases}$$

Note that A is TU so $\begin{bmatrix} A^T & I \\ 0 & I \end{bmatrix}$ is TU. So
$\left\{\begin{pmatrix} y \\ z \end{pmatrix} : \begin{bmatrix} A^T & I \\ 0 & I \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \ge \begin{bmatrix} c \\ 0 \end{bmatrix}\right\}$
is an integral polyhedron.

Moreover, since $\begin{bmatrix} c \\ 0 \end{bmatrix}$ is (0,1)-valued, each extreme point is (0,1)-valued.
(See assignment 5)
Hence there is an optimal solution $(\bar{y}, \bar{z})$ to (D) that is $(0, \pm 1)$ valued.

Deleted from following lecture:
Since $\bar{z} \ge 0$, $\bar{z}$ is (0,1) valued.
Note that z_e is in two constraints:
$\bar{y}_v - \bar{y}_u + \bar{z}_e \ge c_e$ and $\bar{z}_e \ge 0$
So since $u_e > 0$, $\bar{z}_e = \max(0, c_e - \bar{y}_v + \bar{y}_u)$

Note that, for any $\alpha \in \mathbb{R}$, $(\bar{y}, -\alpha, \bar{z})$ is an optimal solution for (D), so we may assume that $\bar{y}_t = 0$.
Let $S = \{v \in V : \bar{y}_v \ge 1\}$ and $T = \{v \in V : \bar{y}_v \le 0\}$

For each $e = uv \in E$, $\bar{x}_e$ occurs in only two constraints $\bar{y}_v - \bar{y}_u + \bar{z}_e \ge c_e$, $\bar{z}_e \ge 0$

Moreover, $u_e > 0$ so $\bar{z}_e = \max(0, c_e - \bar{y}_v + \bar{u}_u)$

Claim 1

(S,T) is an (s,t)-cut

Proof

(S,T) is clearly a partition and $t \in T$ (since $\bar{y}_t = 0$).
Consider $f = ts$. Since $u_f = \infty$, $\bar{x}_f = 0$ so $0 \ge c_f - \bar{y}_s + \bar{y}_t = 1 - \bar{y}_s$
Then $\bar{y}_s \ge 1$ and $s \in S$ as required.

Claim 2

If $e = uv$, $u \in S$ and $v \in T$ then $\bar{z}_e \ge 1$

Proof

$\bar{z}_e = \max(0, c_e - \bar{y}_v + \bar{y}_u) \ge 1$ since $\bar{y}_v \le 0$ and $\bar{y}_u \ge 1$

By claim 2,
$u(S,T) \leq u^t \bar{z} = OPT(D) = OPT(P) =$ max value of a feasible (s,t)-flow
However,
$U(S,T) \geq$ max value of a feasible (s,t)-flow
by the earlier claim ■

Complexity Theory

November-09-12 2:57 PM

Decision Problems
A **decision problem** is a yes/no question on a countable set of instances; the "size" of an instance is the length of a binary encoding.

Polynomial-Time
An algorithm is polynomial time if its running time is bounded by a polynomial in the size of the input. P is the set of all decision problems that can be shown to be polynomial time.

Claim
LP feasibility is in NP

Exercise
Show that IP is in NP

Find a rational half-line

Nondeterministic Polynomial time
A decision problem P is in NP if there is a polynomial time algorithm A and a polynomial p such that
1) for each yes-instance I of P there is a certificate C such that $|C| \leq p(|I|)$, and A accepts (I, C)
2) for each instance I of P and any C with $|C| \leq p(|I|)$, A rejects (I, C).

If $(NOT\ P) \in NP$, then P is in co-NP

Claim
LP feasibility is in co-NP

Major Conjectures
1) $P = NP$
2) $NP =$ co-NP
3) $P = NP \cap$ co-NP

Polynomial Time Reduction
We say that P_1 reduces to P_2 if there is a polynomial time algorithm A such that for each instance I_1 of P_1, A generates an instance I_2 of P_2 such that I_1 is a yes instance of P_1 iff I_2 is a yes instance of P_2.

NP-Completeness
A problem $P \in NP$ is NP-Complete if every problem in NP reduces to it.

Theorem (Cook)
IP feasibility is NP-complete
(Cook used "3-SAT")

Formalism
Let $A = \{0, 1, -\}$ (can fix any finite alphabet with $|A| \geq 2$)
A* denotes the set of finite words in A

Problem
A problem is any subset of A* (these are the yes-instances).

Given $w_1, w \in A^*$ we say that w **contains** w_1 iff $w = \alpha w_1 \beta$.
We say that w' is obtained from w by replacing w_1 by w_2 if $w = \alpha w_1 \beta$ and $w' = \alpha w_2 \beta$
(not we are only replace on instance).

Algorithm
An **algorithm** is a sequence
$(w_1, w_1'), \dots, (w_k, w_k')$

Start: w
Step 1: For $i = 1, \dots, k$ if w contains w_i, then replace the first instance of w_i with $w_i{}'$. Repeat from Step 1.
Step 2: Stop

An algorithm solves a problem Π if Π is the set of instances on which the algorithm terminates.

LP Feasibility Problems
Instance: $A \in \mathbb{Q}^{m \times n},\ b \in \mathbb{Q}^n$
Question: Does there exist $x \in \mathbb{Q}^n$ such that $Ax \leq b$?

IP Feasibility
Instance: $A \in \mathbb{Q}^{m \times n},\ b \in \mathbb{Q}^m$
Question: Does there exist $z \in \mathbb{Z}^n$ such that $Ax \leq b$

Bipartite Matching Problem
Instance: G bipartite, $k \in \mathbb{Z}_+$
Question: Does G have a matching of size $\geq k$?

Clique Problem
Instance: A graph G, $k \in \mathbb{Z}_+$
Question: Does G contain a set of k pairwise adjacent vertices?

Examples of NP problems
1) Bipartite matching problem
   - Give the matching
2) Clique problem
   - Give the clique
3) LP feasibility problems
   - Give a solution, whose size is small

Proof of Claim: LP feasibility is in NP
Consider $P = \{x \in \mathbb{R}^n : Ax \leq b\}$
Suppose that $\bar{x} \in$ P
We may assume without loss of generality that $\bar{x} \geq 0$
(by changing variables. This does not change the size of the solution)
By adding some inequalities we may assume that $P \subseteq \mathbb{R}^n_+$

Now P has an extreme point $\tilde{x}$.
there is a subsystem $A'x \leq b'$ of $Ax \leq b$ such that A' is $n \times n$, $rank(A') = n$ and $A'\tilde{x} = b'$
Now $\tilde{x}$ is the unique solution to $A'x = b'$ and we have shown that
$size(\tilde{x}) \leq \text{polynomial}\big(size(A', b')\big)$
■

Example
Bipartite Matching is in co-NP
Show a vertex cover of size $< k$

Proof of Claim: LP feasibility ∈ co-NP
By the Farkas Lemma,
$(Ax \leq b)$ is infeasible, iff $(A^T y \geq 0, b^T y \leq -1, y \geq 0)$ is feasible.
This "reduces" NOT(LP feasibility) to an instance of LP feasibility of a similar size ■

Example
Consider the clique problem on an instance $G = (V, E),\ k$
Construct an instance of IP feasibility

$$\begin{cases} \sum_{v \in V} x_v = k & \\ x_u + x_v \leq 1, & u, v \text{ non-adjacent} \\ 0 \leq x \leq 1 & \\ x \text{ integer} & \end{cases}$$

This reduces the Clique Problem to IP feasibility.

Sketch of Proof of Cook's Theorem
Consider a problem $P \in NP$.
Let A, p be the requisite algorithm and polynomial.
Start with input I, certificate C, and empty memory
Algorithm to verify (I, C) runs in polynomial time with polynomial memory.
Consider all bits in memory at each time step. Each unknown one is a $(0, 1)$ variable.
Linear inequalities are used to describe feasible transitions according to the model of computation, and to describe the accepting state.

Now this is a IP feasibility problem.

Exercise
Write an algorithm for checking $a + b = c$ on given integers a, b, c.

An algorithm is **polynomial-time** if there is a polynomial P such that for each instance I on which the algorithm terminates, the algorithm termination in $\leq p\big(length(I)\big)$ steps.

Nonlinear Optimization

November-16-12 2:32 PM

Nonlinear Optimization Problem

minimize$(f(x) : x \in S)$
$S \subseteq \mathbb{R}^n, \quad f: S \to \mathbb{R}$

Inf / Sup

$\inf\{f(x): x \in S\} = \max_{z \in [-\infty,\infty]} (f(x) \geq z : x \in S)$
$\sup\{ f(x): x \in S\} = \min_{z \in [-\infty,\infty]} (f(x) \leq z : x \in S)$

Compactness

A set $S \subseteq \mathbb{R}^n$ is **closed** if for each convergent sequence $(x^{(1)}, x^{(2)}, \dots)$ in S, $\quad \lim_{i\to\infty} x^{(i)} \in S$

For $c \in \mathbb{R}^n, r \in \mathbb{R}$ define
$\text{Ball}(c, r) = \{x \in \mathbb{R}^n : \|x - c\| \leq r\}$

S is **bounded** if there exists $d \in \mathbb{R}$ such that $S \subseteq \text{Ball}(0, d)$

S is **compact** if S is closed and bounded.

Bolzano-Weierstrass Theorem

If $S \subseteq \mathbb{R}^n$ is a compact set, then any sequence of points in S contains a convergent subsequence.

Corollary (Weierstrass)

If $S \subseteq \mathbb{R}^n$ is nonempty and compact and $f: S \to \mathbb{R}$ is continuous, then there exists $x \in S$ minimizing f.

Nearest Point

Let $S \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^n$.
We call $\bar{x} \in$ S a **nearest point** to z if $\|\bar{x} - z\| \leq \|x - z\|$ for all $x \in S$

Theorem

If $S \subseteq \mathbb{R}^n$ is nonempty and closed and $z \in \mathbb{R}^n$, then there exists a nearest point in S to z.

Separating Hyperplane Theorem

Let $S \subseteq \mathbb{R}^n$ be a closed convex set and $z \subseteq \mathbb{R}^n$.
Then $z \notin S$ iff there exists $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^T x \leq b$ for all $x \in S$ and $a^T z > b$.

Boundary of a Set

A point $x \in S$ is in the interior of S if there exists $\epsilon > 0$ such that $\text{Ball}(x, \epsilon) \subseteq S$
The **interior**, denoted $\text{int}(S)$ is the set of all interior points.
The **boundary** is the set $S - \text{int}(S)$.

Theorem (3)

Let $S \subseteq \mathbb{R}^n$ be a closed convex set and let $\bar{x} \in$ S be on the boundary. Then there exists a nonzero $c \in \mathbb{R}^n$ such that $\bar{x}$ minimizes $(c^T x : x \in S)$

minimize$(f(x) : x \in S)$
$S \subseteq \mathbb{R}^n, \quad f: S \to \mathbb{R}$

Recall that

1) "in theory" we can reduce to the case that $f(x)$ is linear and S is convex
   By rewriting as
   $\min \left(y : \binom{y}{x} \in \left\{ \binom{f(x)}{x} : x \in S \right\} \right)$
   and can replace $\left\{ \binom{f(x)}{x} : x \in S \right\}$ by its convex hull
2) Small problems can be nontrivial
   e.g.
   $$\begin{cases} \text{minimize} & \sin(\pi x)^2 + \sin(\pi y)^2 + \sin(\pi z)^2 \\ \text{subject to} & x^3 + y^3 = z^3 \\ & x, y, z \geq 1 \end{cases}$$

Note that $\min(f(x): x \in S)$ may not be attained even if $S \neq \emptyset$ and $f(x)$ is bounded below.
Eg. $\min\{x: x > 0\}$

Proof of Theorem

Let $\bar{x} \in$ S and $d = \|\bar{x} - z\|$.
Now $S \cap \text{Ball}(z, d)$ is compact and nonempty. By the corollary to B-W Theorem, there is a nearest point $\tilde{x} \in \text{S} \cap \text{Ball(z, d)}$ to z. Clearly $\tilde{x}$ is a nearest point in S to z.

■

Exercises

1) Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed, convex set and let $z \in \mathbb{R}^n$. Prove that there is a unique nearest point in S to z.
2) Let $z, a, b \in \mathbb{R}^n$ and let $S = conv(\{a, b\})$
   Prove that a is the nearest point in S to z iff $(z - a)^T (b - a) \leq 0$

Proof of Separating Hyperplane Theorem

$\Leftarrow$ Trivial
$\Rightarrow$ Suppose that $z \notin S$. We may assume that $S \neq \emptyset$
So, by the previous theorem, there is a nearest point $\bar{x}$ to z in S.
Let $a = z - \bar{x}$ and $b = a^T \bar{x}$
Now $a^T z = a^T (z - \bar{x}) + a^T \bar{x} = \|z - \bar{x}\|^2 + b > b$

Let $x \in S$ and let $L = conv(\{x, \bar{x}\})$ Since $L \subseteq S$, $\bar{x}$ is the nearest point in L to z. Now, by Exercise 2, $(z - \bar{x})^T (x - \bar{x}) \leq 0$
That is, $a^T x \leq b$

Exercise

Let $S \subseteq \mathbb{R}^n$ be a closed convex set and let $\bar{x} \in$ S be in the boundary.
Prove that there exists $\bar{z} \notin$ S such that $\bar{z}$ is the nearest point in S to z.

Proof Sketch of Theorem (3)

Take $z \notin S$ such that $\bar{x}$ is the nearest point in S to z. Let $c = \bar{x} - z$.
Now continuous as in the proof of the separating Hyperplane Theorem. ■

Certifying Optimality

November-19-12 2:40 PM

Cost Splitting Theorem

(Sufficient condition for Optimality)
Let $S_1, \dots, S_m \subseteq \mathbb{R}^n$, $let\ S = S_1 \cap \dots \cap S_m$, let $c \in \mathbb{R}^n$, and let $\bar{x} \in S$.
If there exist $c_1, \dots, c_m \in \mathbb{R}^n$ such that $c = c_1 + \dots + c_m$ and such that $\bar{x}$ minimizes each $(c_i^T x : x \in S_i)$ for $i = 1, \dots, m$ then x minimizes $\{c^T x : x \in S\}$

Cost Splitting for Linear Programming

Let $c, \alpha_1, \dots, \alpha_m \in \mathbb{R}^n$ and $b_1, \dots, b_m \in \mathbb{R}$
Now let $H_i = \{x \in \mathbb{R}^n : (\alpha_i)^T x \geq b_i\}$
Consider the linear program
$(P) \min(c^T x\ : x \in H_1 \cap \dots \cap H_m)$
Let $\bar{x} \in \mathrm{H}_1 \cap \dots \cap \mathrm{H}_\mathrm{m}$

Problem

Do there exist $c_1, \dots, c_m \in \mathbb{R}^m$ such that $c = c_1 + \dots + c_m$ and such that $\bar{x}$ minimizes each of $(c_i^T x\ :\ x \in H_i)$

Let $i \in \{1, \dots, m\}$
Now $\bar{x}$ minimizes $(c_i^T x : x \in H_i)$ iff

I) $c_i = 0$ or;
II) $(\alpha_i)^T \bar{x} = b_i$ and $c_i = \lambda_i \alpha_i$ for some $\lambda_i \geq 0$

Define $I = \{i \in \{1, \dots, m\} : \alpha_i^T \bar{x} = b_i\}$
These give the "tight constraints"
We want to find $(\lambda_i \geq 0 : i \in I)$ such that

$$c = \sum_{i \in I} \lambda_i \alpha_i$$

Theorem

$\bar{x}$ is optimal for (P) iff $c \in \mathrm{cone}(\alpha_i : i \in I)$
That is, for LP the cost splitting theorem is necessary and sufficient.

Strong Cost Splitting Theorem

(Necessary and Sufficient Conditions for optimality in convex optimization)
Let $S_1, \dots, S_m \subseteq \mathbb{R}^n$ be closed convex sets with $\mathrm{int}(S_1 \cap \dots \cap S_n) \neq \emptyset$.
Let $c \in \mathbb{R}^n$ and $\bar{x} \in S_1 \cap \dots \cap S_m$
Then $\bar{x}$ minimizes $(c^T x : x \in S_1 \cap \dots \cap S_m)$ iff there exists $c_1, \dots, c_m \in \mathbb{R}^n$ such that $c = c_1 + \dots + c_m$ and $\bar{x}$ minimizes each of $(c_i^T x : x \in S_i)$, $i = 1, \dots, m$

Remarks

1) Let $S \subseteq \mathbb{R}^n$, $n \geq 2$ be a convex set. $Int(S) = \emptyset$ iff there exists a hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $S \subseteq H$
2) Let F be the affine subspace spanned by $S_1 \cap \dots \cap S_m$ and let $\tilde{F} = \{x - \bar{x} : x \in F\}$. If $c \in \tilde{F}$ then we don't need the condition that $\mathrm{int}(S_1 \cap \dots \cap S_m) \neq \emptyset$.
   If $c \notin \tilde{F}$ we can take the orthogonal projection of c onto $\tilde{F}$.
3) Using (2) it is a straightforward exercise to deduce the Strong Duality Theorem from the Strong Cost Splitting Theorem.

Closed Sets

Note that the intersection of closed sets is closed.
Hence, if $S \subseteq \mathbb{R}^n$, there is a unique minimal closed set containing S.
This set is denoted closure(S).

Certifying Optimality

Problem: How can we prove that $\bar{x} \in \mathrm{S}$ minimizes $(c^T x : x \in S)$?

Answer: In general, we can't.
Nonlinear programming is undecidable.

Proof of Cost Splitting Theorem

$c^T \bar{x} \geq \min(c^T x : x \in S) = \min(c_1^T x + \dots + c_m^T x : x \in S)$
$\geq \min(c_1^T x_1 + \dots + c_m^T x_m : x_1, \dots, x_m \in S)$
$= \min(c_1^T x_1 : x_1 \in S) + \dots + \min(c_m^T x_m : x_m \in S)$
$\geq \min(c_1^T x_1 : x_1 \in S_1) + \dots + \min(c_m^T x_m : x_m \in S_m) = c_1^T \bar{x} + \dots + c_m^T \bar{x} = c^T \bar{x}$
■

Remark

Cost splitting is not always possible

Example

$S_1 = \mathrm{Ball}\left(\begin{bmatrix}0\\0\end{bmatrix}, 1\right)$, $\quad S_2 = \mathrm{Ball}\left(\begin{bmatrix}2\\0\end{bmatrix}, 1\right)$

$S = S_1 \cap S_2 = \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} = \{\bar{x}\}$

$c = \begin{bmatrix}1\\1\end{bmatrix}$

$\bar{x}$ minimizes $(c^T x : x \in S)$

$\bar{x}$ minimizes $(c_1 x : x \in S_1)$ iff $c_1 = \begin{bmatrix}-a\\0\end{bmatrix}$ $a \geq 0$

$\bar{x}$ minimizes $(c_2 x : x \in S_2)$ iff $c_2 = \begin{bmatrix}b\\0\end{bmatrix}$, $b \geq 0$

So $c \neq c_1 + c_2$ for any $a, b \geq 0$

Proof of Theorem

$\Leftarrow$ Cost splitting Theorem
$\Rightarrow$ Suppose that $\bar{x}$ is optimal for (P).
Now rewrite (P) as

$$\begin{cases} \text{mim} & c^T x \\ \text{subject to} & \alpha_i^T x \geq b_i \quad i = 1, \dots, m \end{cases}$$

The dual is

$$(D) \begin{cases} \text{max} & b_1 y_1 + \dots + b_m y_m \\ \text{subject to} & \alpha_1 y_1 + \dots + \alpha_m y_m = c \\ & y_1, \dots, y_m \geq 0 \end{cases}$$

Let $\bar{y}$ be optimal for (D).
By the complementary slackness conditions, for $i \in \{1, \dots, m\} - I$ we have $\bar{y}_i = 0$. So $c \in \mathrm{cone}(\alpha_i : i \in I)$ as required. ■

Strong Cost Splitting Theorem

November-23-12 2:32 PM

Strong Cost Splitting Theorem

(Necessary and Sufficient Conditions for optimality in convex optimization)
Let $S_1, \dots, S_m \subseteq \mathbb{R}^n$ be closed convex sets with $int(S_1 \cap \dots \cap S_n) \neq \emptyset$.
Let $c \in \mathbb{R}^n$ and $\bar{x} \in S_1 \cap \dots \cap S_m$
Then $\bar{x}$ minimizes $(c^T x : x \in S_1 \cap \dots \cap S_m)$ iff there exists $c_1, \dots, c_m \in \mathbb{R}^n$ such that $c = c_1 + \dots + c_m$ and $\bar{x}$ minimizes each of $(\, c_i^T x : x \in S_i),\ \ i = 1, \dots, m$

Lemma 1

If $S_1, S_2 \subseteq \mathbb{R}^n$ are convex and $int(S_1 \cap S_2) \neq \emptyset$ then
$closure(S_1 \cap S_2) = closure(S_1) \cap closure(S_2)$

Tangent Cone

For $S \subseteq \mathbb{R}^n$ and $\bar{x} \in S$, $T(\bar{x}, S) = \text{closure}(\text{cone}(\{x - \bar{x} : x \in S\}))$

Remarks

1) If S is convex, then $T(\bar{x}, S)$ is a closed convex cone.
2) This definition is nonstandard but agrees with the usual definition on convex sets.

Theorem 2

Let $S_1, S_2 \subseteq \mathbb{R}^n$ be closed convex sets with $int(S_1 \cap S_2) \neq \emptyset$ and let $\bar{x} \in S_1 \cap S_2$.
Then $T(\bar{x}, S_1 \cap S_2) = T(\bar{x}, S_1) \cap T(\bar{x}, S_2)$

Note

We need the condition $int(S_1 \cap S_2) \neq \emptyset$
Consider the example of the two balls.

Convex Cones

Separating Hyperplane Theorem for Cones

Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and $z \in \mathbb{R}^n$. If $z \notin K$ then there exists $c \in \mathbb{R}^n$ such that $c^T x \geq 0\ \forall\ x \in K$ and $c^T z < 0$.

Duality for Cones

For $S \subseteq \mathbb{R}^n$, define $S^* = \{c \in \mathbb{R}^n : c^T x \geq 0,\ x \in S\}$
$S^* = \{c \in \mathbb{R}^n : c^T x \geq 0, \quad \forall\ x \in S\}$

Remarks

1) If $0 \in S$, then S^* is the set of all $c \in \mathbb{R}^n$ such that
   0 minimizes $(c^T x : x \in S)$
   In other words, S^* is the set of all directions that are minimized by 0 over S
2) If K is a cone, then K^* is called the dual of K.

Lemma 3

For any $S \subseteq \mathbb{R}^n$, S^* is a closed convex cone.

Lemma 4

Let $S \subseteq \mathbb{R}^n$ and let K be the smallest closed convex cone containing S. Then $S^{**} = K$

Duality Theorem For Cones

If $K \subseteq \mathbb{R}^n$ is a closed convex cone, then $K^{**} = K$

Normal Cone

Let $S \subseteq \mathbb{R}^n$ and $\bar{x} \in$ S.
$N(\bar{x}, S) = \{c \in \mathbb{R}^n : c^T(x - \bar{x}) \geq 0, \quad x \in S\}$

Note that:
1) $c \in N(\bar{x}, S)$ iff $\bar{x}$ minimizes $(c^T x : x \in S)$.
2) $N(\bar{x}, S) = \{x - \bar{x} : x \in S\}^*$

Lemma 5

Let $S \subseteq \mathbb{R}^n$ be a convex set and $\bar{x} \in$ S. Then
$N(\bar{x}, S) = T(\bar{x}, S)^*$

Sum of Sets

For $S_1, S_2 \subseteq \mathbb{R}^n$, let
$S_1 + S_2 = \{a + b : a \in S_1, b \in S_2\}$

- [ ] **Exercise**

Let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex cones. Prove that $K_1 + K_2$ is the smallest convex cone containing K_1 and K_2

Remark

There exist closed sets $S_1, S_2 \subseteq \mathbb{R}^n$ such that $S_1 + S_2$ is closed

Lemma 6

Proof of Lemma 1

- [ ] Exercise

Example of Tangent Cone

$S = \text{Ball}\left(\begin{bmatrix}0\\0\end{bmatrix}, 1\right), \qquad \bar{x} = \begin{bmatrix}-1\\0\end{bmatrix}$
$\{x - \bar{x} : x \in S\} = \text{Ball}\left(\begin{bmatrix}1\\0\end{bmatrix}, 1\right)$
$cone\left(\text{Ball}\left(\begin{bmatrix}1\\0\end{bmatrix}, 1\right)\right) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} \cup \{x \in \mathbb{R}^2 : x_1 > 0\}$
$closure\left(cone\left(\text{Ball}\left(\begin{bmatrix}1\\0\end{bmatrix}, 1\right)\right)\right) = \{x \in \mathbb{R}^2 : x_1 \geq 0\}$

Proof of Theorem 2

We can translate S_1 and S_2 so that $\bar{x} = 0$.
Now since S_1 and S_2 are convex and $0 \in S_1 \cap S_2$
$\quad cone(S_1 \cap S_2) = cone(S_1) \cap cone(S_2)$
$\quad$(Consider scaling points in one set into the other)
Since $int(S_1 \cap S_2) \neq \emptyset$, $int\big(cone(S_1) \cap cone(S_2)\big) \neq \emptyset$
So by Lemma 1,
$T(\bar{x}, S_1 \cap S_2) = closure\big(cone(S_1 \cap S_2)\big) = closure\big(cone(S_1) \cap cone(S_2)\big)$
$= closure\big(cone(S_1)\big) \cap closure\big(cone(S_2)\big) = T(\bar{x}, S_1) \cap T(\bar{x}, S_2)$
■

Proof of Separating Hyperplane Theorem for Cones

By the Separating Hyperplane Theorem, there exists $c \in \mathbb{R}^n$and $b \in \mathbb{R}$ such that $c^T x \geq b$ and $c^T z < b$.
We may assume that $b = \inf(c^T x : x \in K)$
Note that $0 \in K$ so $b \leq 0$

We may assume that $b < 0$ (otherwise we're done)
There exists $\tilde{x} \in$ K with $c^T \tilde{x} < \frac{b}{2}$
However, since K is a cone $2\tilde{x} \in$ K and $c^T(2\tilde{x}) < b$ - contradiction ■

Proof of Lemma 3

For each $x \in S$, let $H(x) = \{c \in \mathbb{R}^n : c^T x \geq 0\}$
So $H(x)$ a closed convex cone. Now
$S^* = \bigcap_{x \in S} H(x)$
which is also a closed convex cone. ■

Proof of Lemma 4

By definition, $S \subseteq S^{**}$ and, by Lemma 3, S^{**} is a closed convex cone.
So $K \subseteq S^{**}$
Suppose that $K \neq S^{**}$, then $S^{**} \neq \emptyset$ and let $z \in S^{**} \setminus K$
By the separating Hyperplane Theorem, there exists $c \in \mathbb{R}^n$ such that $c^T x \geq 0$ for all $x \in K$ and $c^T z < 0$.
Since $S \subseteq K,\ c \in S^*$
But $c^T z < 0,\ so\ z \notin S^{**}$ - contradiction . ■

Proof of Duality Theorem for Cones

Immediate by Lemma 4

Proof of Lemma 5

$\tilde{S} = \{x - \bar{x} : x \in S\}$
By definition,
$N(\bar{x}, S) = \tilde{S}^*$ and $T(\bar{x}, S) = \text{closure}\left(\text{cone}(\tilde{S})\right)$
By Lemma 4
$T(\bar{x}, S) = \tilde{S}^{**} = \big(N(\bar{x}, S)\big)^*$
By the duality theorem, $T(\bar{x}, S)^* = N(\bar{x}, S)$ ■

Example of Remark

$S_1 = \{x \in \mathbb{R}^2 : x_2 \geq e^{x_1}\}$
$S_2 = \{x \in \mathbb{R}^2 : x_2 = 0\}$
$S_1 + S_2 = \{x \in \mathbb{R}^2 : x_2 > 0\}$

Proof of Lemma 6

See assignment 6

Proof of Lemma 7

Let $S = K \cap \{x \in \mathbb{R}^n : d^T x = 1\}$
Since S is the intersection of two closed sets, S is closed.
Since $d \in int(K^*)$, for each nonzero $x \in K,\ d^T x > 0$.
Hence $K = cone(S)$

Chose $\epsilon > 0$ such that $\text{Ball}(d, \epsilon) \subseteq Int(K^*)$
Consider any $c \in \mathbb{R}^n$ with $\|c\| = \epsilon$
Now $d - c \in K^*$ so for any $x \in S$
$0 \leq (d - c)^T x = d^T x - c^T x = 1 - c^T x$
Thus $c^T x \leq 1$. It follows that S is bounded.

Remark

There exist closed sets $S_1, S_2 \subseteq \mathbb{R}^n$ such that $S_1 + S_2$ is closed

Lemma 6

If $S_1, S_2 \subseteq \mathbb{R}^n$ are compact, then $S_1 + S_2$ is compact.

Lemma 7

If $K \subseteq \mathbb{R}^n$ is a closed cone and $d \in int(K^*)$, then $\{x \in K : d^T x = 1\}$ is compact and $K = cone(\{x \in K : d^T x = 1\})$

Lemma 8

If $K_1, K_2 \in \mathbb{R}^n$ are closed cones with $int(K_1^* \cap K_2^*) \neq \emptyset$ then $K_1 + K_2$ is closed

Theorem ($m = 2$)

Let $S_1, S_2 \subseteq \mathbb{R}^n$ be closed convex sets with $int(S_1 \cap S_2) \neq \emptyset$.
Then $N(\bar{x}, S_1 \cap S_2) = N(\bar{x}, S_1) + N(\bar{x}, S_2)$

Theorem

Let $S_1, S_2 \subseteq \mathbb{R}^n$ be closed convex sets and $\bar{x} \in S_1 \cap S_2$ such that
1) $T(\bar{x}, S_1 \cap S_2) = T(\bar{x}, S_1) \cap T(\bar{x}, S_2)$ and
2) $N(\bar{x}, S_1) + N(\bar{x}, S_2)$ is closed

then $N(\bar{x}, S_1 \cap S_2) = N(\bar{x}, S_1) + N(\bar{x}, S_2)$

Has essentially the same proof as the above.

Claim

If S_1 and S_2 are polyhedra then (1) and (2) are satisfied.

Consider any $c \in \mathbb{R}^n$ with $\|c\| = \epsilon$
Now $d - c \in K^*$ so for any $x \in S$
$0 \leq (d - c)^T x = d^T x - c^T x = 1 - c^T x$
Thus $c^T x \leq 1$. It follows that S is bounded.

Proof of Lemma 8

Let $d \in int(K_1^* \cap K_2^*)$,
$S_1 = K_1 \cap \{x \in \mathbb{R}^n : d^T x = 1\}$ and
$S_2 = K_2 \cap \{x \in \mathbb{R}^n : d^T x = 1\}$

By Lemma 7, S_1 and S_2 are compact, so, by Lemma 6, $S_1 + S_2$ is compact.
Now $K_1 + K_2 = cone(S_1 + S_2)$ so $K_1 + K_2$ is closed. ∎

★ Grand Finale *Jazz Hands*

Recall that $N(\bar{x}, S)$ is the set of all $c \in \mathbb{R}^n$ such that $\bar{x}$ minimizes $(c^T x : x \in S)$

Cost-Splitting Theorem (Reworded)

If $S_1, \dots, S_m \subseteq \mathbb{R}^n$ and $\bar{x} \in S_1 \cap \dots \cap S_m$ then
$N(\bar{x}, S_1) + \dots + N(\bar{x}, S_m) \subseteq N(\bar{x}, S_1 \cap \dots \cap S_m)$

Strong Cost-Splitting Theorem

If $S_1, \dots, S_m \subseteq \mathbb{R}^n$ are closed convex sets and $int(S_1 \cap \dots \cap S_m) \neq \emptyset$, then
$N(\bar{x}, S_1) + \dots + n(\bar{x}, S_m) = N(\bar{x}, S_1 \cap \dots \cap S_m)$

Note that both of these results follow from the special case that $m = 2$

Proof of Theorem ($m = 2$)

By the Cost Splitting Theorem, $N(\bar{x}, S_1) + N(\bar{x}, S_2) \subseteq N(\bar{x}, S_1 \cap S_2)$

If equality does not hold, there exists $z \in N(\bar{x}, S_1 \cap S_2) \setminus \big(N(\bar{x}, S_1) + N(\bar{x}, S_2)\big)$
Since $int(S_1 \cap S_2) \neq \emptyset$, $int\big(T(\bar{x}, S_1) \cap T(\bar{x}, S_2)\big) \neq \emptyset$
So by Lemmas 5 and 8, $N(\bar{x}, S_1) + N(\bar{x}, S_2)$ is closed.

So by the Separating Hyperplane Theorem, there exists $d \in \mathbb{R}^n$ such that
1) $d^T x \geq 0$ for all $x \in N(\bar{x}, S_1) + N(\bar{x}, S_2)$ and
2) $d^T z < 0$

By (1) and Lemma 5, $d \in T(\bar{x}, S_1) \cap T(\bar{x}, S_2)$
By Theorem 2,
$d \in T(\bar{x}, S_1 \cap S_2)$

However, $z \in N(\bar{x},\ S_1 \cap S_2)$ and $d^T z < 0$ contradicting Lemma 5. ∎

Proof of Claim

1)
Recall that for a convex set S, $T(\bar{x}, S) = closure\big(cone(\{x - \bar{x} : x \in S\})\big)$
Note that $cone(\{x - \bar{x} : x \in S_1 \cap S_2\}) = cone(\{x - \bar{x} : x \in S_1\}) \cap cone(\{x - \bar{x} : x \in S_2\})$

So if $cone(\{x - \bar{x} : x \in S_1\})$ and $cone(\{x - \bar{x} : x \in S_2\})$ are closed then (1) holds

If S_1 and S_2 are polyhedra then $cone(\{x - \bar{x} : x \in S_1\})$ and $cone(\{x - \bar{x} : x \in S_2\})$ are polyhedral cones and hence are closed. So (1) holds.

2)
The dual of a polyhedral cone is a polyhedral cone (why?)
So $N(\bar{x}, S_1)$ and $N(\bar{x}, S_2)$ are polyhedral cones.
The sum of polyhedral cones is a polyhedral cone. So $N(\bar{x}, S_1) + N(\bar{x}, S_2)$ is closed. Hence (2) is satisfied.

Exercise

- [ ] Show that the above theorem and claim imply the Strong Duality Theorem

For $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ have LP and dual:
(P) $\max(c^T x : Ax \leq b)$
(D) $\min(b^T y : A^T y = c, \quad y \geq 0)$
Define $S_P, S_D \subseteq \mathbb{R}^{n+m}$ by
$S_P = \{(x, y) \in \mathbb{R}^{n+m} : Ax \leq b\}$
$S_D = \{(x, y) \in \mathbb{R}^{n+m} : A^T y = c, \quad y \geq 0\}$
Define $a = -c + b$ and consider the problem
(Q) $\min(a^T(x, y) : (x, y) \in S_P \cap S_D)$

Suppose that $\tilde{x}$ is optimal for (P) and $\tilde{y}$ is optimal for (D). Note that
$(\tilde{x}, \tilde{y}) = \min(-c^T x : (x, y) \in S_1) = \min(b^T y : (x, y) \in S_2)$
S_1 and S_2 are polyhedra so they satisfy (1) and (2) in the alternate Strong Duality Theorem and hence $(\tilde{x}, \tilde{y})$ is optimal for (Q).

still have to prove optimal value is 0

Convex Optimization Algorithms

November-30-12 2:32 PM

$(P)\ \min(c^T x : x \in S)$
$S \subseteq \mathbb{R}^n$ closed, convex

Assumptions

1) $S \subseteq \text{Ball}(0, R)$, R given
2) S is given by a "separation oracle"
   Given $z \in \mathbb{R}^n$, the oracle returns
   yes: if $z \in S$
   a separating hyperplane if $z \notin S$

Remarks

Consider a linear program
$(P)\ \min(c^T x : Ax \geq b)$
for $z \notin \{x \in \mathbb{R}^n : Ax \geq b\}$ it is trivial to get a separating hyperplane. There exists R with $size(R) \leq poly(size(A, b))$ such that solving $\min(c^T x : x \in P \cap \text{Ball}(0, R))$ solves (P).

Feasibility Problems

Given a closed convex set $S \subseteq \mathbb{R}^n$, is S nonempty?

Note that:

1) If we can solve the feasibility problem, then we can solve (P)
   Consider $S \cap \{x \in \mathbb{R}^n : c^T x \leq \alpha\}$ and use binary search on α.
2) For a linear program $(P)\ \min(c^T x : Ax \geq b)$
   consider the feasibility problem:
   $(Ax \geq b, \quad A^T y = c, \quad y \geq 0, \quad b^T y = c^T x)$
   If $(x, \bar{y})$ satisfies (F) then $\bar{x}$ is optimal for (P)

Ellipsoid Method

Method to Solve Feasibility Problem

Ellipsoid

An ellipsoid is an affine transformation of Ball$(0, 1)$
For $c \in \mathbb{R}^n$ and a symmetric positive definite matrix $D \subseteq \mathbb{R}^{n \times n}$ we define $ell(c, D) = \{x \in \mathbb{R}^n : (x - c)^T D^{-1} (x - c) \leq 1\}$

Positive Definite

A matrix A is positive definite iff $x^T A x > 0\ \forall x \neq 0$

Useful Facts

1) $vol(ell(c, D)) = \frac{4}{3}\pi\sqrt{\det(D)}$
2) If $a \in \mathbb{R}^n$ and $S = ell(c, D) \cap \{x \in \mathbb{R}^n : a^T x \geq a^T c\}$
   then there is an ellipsoid $ell(c', D')$ such that $S \subseteq ell(c', D')$ and
   $\frac{vol(ell(c', D'))}{vol(ell(c, D))} \leq e^{-\frac{1}{2n-2}}$

Idea

Suppose that $S \subseteq ell(c, D)$

If $c \in S$ then S is feasible.
If not, then there exists $a \in \mathbb{R}^n$ such that $S \subseteq \{x \in \mathbb{R}^n : a^T x \geq a^T c\}$.
Find the smallest ellipse $ell(c', D')$ containing $ell(c, D) \cap \{x \in \mathbb{R}^n : a^T x \geq a^T c\}$

For any $\epsilon > 0$, let
$k = (2n - 2)\left\lceil \log\left(\frac{\frac{4\pi}{3}R^n}{\epsilon}\right)\right\rceil$
After k iterations of the above we either

1) find a feasible solution, or
2) prove that $vol(S) \leq \epsilon$

Linear Programming Feasibility

Let $P = \{x \in \mathbb{R}^n : ax \geq b\}$

Facts:

1) If $int(P) \neq \emptyset$, then $vol(P) \geq e^{-poly(size(A,b))}$
   (Uses that fact that for $x^1, \ldots, x^{n+1} \in \mathbb{R}^n$, $\left|\det \begin{bmatrix} 1 & \cdots & \\ x^1 & \cdots & x^{n+1} \end{bmatrix}\right| = vol\left(conv(x^1, \ldots, x^{n+1})\right)$
2) There exists $\delta > 0$ such that $size(\delta) \leq poly\left(size(A, b)\right)$ such that $P \neq \emptyset$ iff $P(\delta) = \{x \in \mathbb{R}^n : Ax \geq b - \delta\} \neq \emptyset$

Remark

The above facts, together with the ellipsoid method prove that linear programming can be solved in polynomial time.
For more details see CO 471 or CO 463

What's next in C&O

Highly recommended:
CO 450 Combinatorial Optimization (Fall)

- Network flows
- Matching
- Matroid optimization
- Travelling Salesman Problems

CO 463 Convex Optimization (Fall)

- Ellipsoid method
- Duality

CO 471 Semidefinite Optimization (Summer)

- Optimization with quadratic constraints
- linear programming theory extends naturally
- applications in graph theory

Recommended
CO 466 Continuous Optimization
CO 452 Integer Programming

Applied Courses
CO 456 Game Theory
CO 454 Scheduling

Shameless advertising
CO 446 Matroid Theory