TIME ANALYSIS OF ALGORITHMS WITHOUT RECURSION

Time efficiency

We like to know the time efficiency of a program for several reasons:

- To get an estimate of how long a program will run. (Waiting for it to finish may not be feasible!)
- To get an estimate of how large an input the program can handle without taking too long.
- To compare the efficiency of different programs for solving the same problem.

We could run timing tests on the program, but we prefer to analyze the efficiency of the *algorithm* — before we've written the program.

Why?

Big differences

We have seen that, for a given problem, one algorithm may be vastly more efficient than others. Examples:

Searching a list of n elements:

- linear search takes time on the order of n.
- binary search takes time on the order of $\log_2 n$.

Finding the cheapest paving in a graph with n nodes:

- brute force takes time on the order of 2^{n^2}.
- greedy algo takes time on the order of n^2.

[Is "on the order of" bothering you? Good — we'll define things properly very soon.]

Small differences

We can sometimes fine tune a given algorithm to make it a little faster.

Example: Linear search with a dummy record is faster than ordinary linear search because there is less work per iteration.

(But *any* kind of linear search takes takes time on the order of n because there are roughly n iterations in the worst case.)

Big differences are the first priority

When analyzing algorithms for a given problem, it makes sense to pay attention to the big differences first.

- Example: We would choose the greedy strategy for graph paving over the brute force strategy.

Only then does it make sense worry about the small differences.

- Example: There is no sense in fine tuning the brute force algorithm. Any effort towards fine tuning should be spent on the greedy algorithm.

What we need

We need a technique for analyzing algorithm efficiency that:

- is precise about what we mean by "on the order of"
- can distinguish the big differences
- (ideally) allows for quick and easy analysis

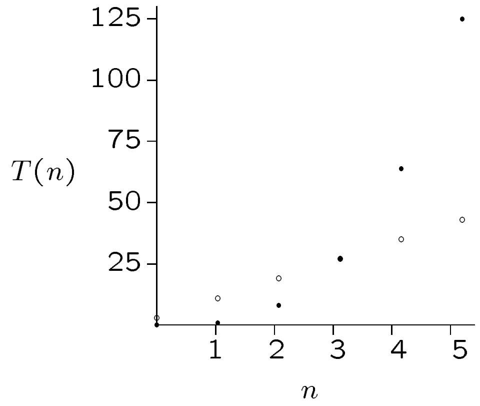
Solution: We will learn a technique that estimates time to within a constant factor.

Because we ignore the constant factors, the analysis is both easier and machine-independent. And it still makes the big distinctions.

Ignoring constant factors

Consider two programs A and B for solving a given problem, with these running times on inputs of size n:

$T_A(n) = n^3$ and $T_B(n) = 8n + 3$.



Which program is faster?

For inputs of size less than 3: program A.
For inputs of size greater than 3: program B.

$n = 3$ is called the **breakpoint**.

B is eventually faster
no matter what the constants

What if program A were a million times faster and program B a million times slower, *i.e.*, if:

$T_A(n) = n^3/1,000,000$ and
$T_B(n) = (8n + 3) * 1,000,000$.

It would still be true that:

- B would eventually be faster than A, and
- B's superiority would grow as n increases.

(The breakpoint would change, however.)

This is true no matter what the constants are!

Conclusion: For large values of n, the form of this mathematical function has more effect on its growth rate than a constant multiple.

Growth rates of various functions

function	Approximate Value of $T(n)$ for $n =$				
$T(n)$	10	100	1,000	10,000	10^5
$\log n$	3	6	9	13	16
$\sqrt{n}$	3	10	31	100	316
n	10	100	1000	10,000	10^5
$n \log n$	30	600	9000	13×10^4	16×10^5
n^2	100	10,000	10^6	10^8	10^{10}
n^3	1000	10^6	10^9	10^{12}	10^{15}
2^n	1024	10^{30}	10^{300}	10^{3000}	$10^{30,000}$

There is a computational cliff when we reach "exponential" functions: ones in which the variable appears in the exponent.

To get a sense of scale:

- there are 10^{43} atoms in the universe
- there have been 10^{17} seconds since the big bang

If we can perform 1 billion operations per second,

- 10^{16} operations take 1 year
- 10^{20} operations take 10,000 years!

Would it help if we could do 100 million billion per second? How about 10^{50}?

Constants Can Matter

Constant factors are insignificant *relative to the form of the mathematical function*.

But they are important in these situations:

- When we've already picked the algorithm to use, and we're ready to fine tune it.
  We may be able to reduce the constants.
- When two algorithms have the same order.
  Considering the constants may reveal that one is faster than the other.
- When the problem size is small.
  We may be below the breakpoint.

Sometimes, we need to know the actual running time of a program, e.g., with real-time systems.

We can determine running time by directly measuring it (on the desired computer and for various sizes of input with various characteristics).

Towards a Definition

Say we have an algorithm (or program) whose running time on an input of size n is $f(n)$.

We don't know what $f(n)$ is, but we want to estimate it to within a constant factor.
Let's call that estimate $g(n)$.

We would be happy with our estimate $g(n)$ even if the relationship between $g(n)$ and $f(n)$ holds only after some point B.

In other words, from B onwards, we want $g(n)$ to estimate an upper bound on $f(n)$, to within a constant factor.

That is, we want there to be some constant factors c and B such that $f(n) \leq c \cdot g(n)$ for all $n \geq B$.

We don't care what these constants are; we just need them to exist.

O Notation, or "big-oh" notation

Consider any 2 functions f, g defined on the nonnegative integers $\mathrm{N} = \{0, 1, 2, \ldots\}$ such that $f(n), g(n) \geq 0$ for all $n \in \mathrm{N}$.

Definition: $f(n)$ is $O(g(n))$ if there exist positive constants c and B such that
$\boxed{f(n) \leq c \cdot g(n) \text{ for all } n \geq B.}$

This means that, to within a constant factor, $f(n)$ grows no faster than $g(n)$.

We pronounce this:

f has order g, or
f is "oh" or "big-oh" of g

2 Key Properties of O Notation

Constant factors disappear

If $d > 0$, then
$df(n)$ is $O(f(n))$ and $f(n)$ is $O(df(n))$.

Examples:
$6n$ and $\frac{n}{2}$ are $O(n)$.
n is $O(29n)$ and $O(642n)$.

Low-order terms disappear

If $\lim_{n\to\infty} \frac{h(n)}{g(n)} = 0$ then $g(n) + h(n)$ is $O(g(n))$.

Examples:
$n^5 + n^3 + 6n^2$ is $O(n^5)$.
$n^2 + n(\log n)^3$ is $O(n^2)$.

Proving a Big-oh Bound

Example

Prove: $6n + 3$ is $O(n)$.

Proof:

- Let $c = 7$ and $B = 3$
  - $6n + 3 \leq 6n + n$ for all $n \geq 3$, so
  - $6n + 3 \leq 7 \cdot n$ for all $n \geq 3$, so
  - $6n + 3 \leq c \cdot n$ for all $n \geq B$.
- So there exist c and B such that $6n + 3 \leq c \cdot n$ for all $n \geq B$.
- So $6n + 3$ is $O(n)$, by definition of big-oh.

Exercise: Prove that the following function is $O(n^3 \cdot n \log n)$:

$f(n) = (6n^3 + n \log n + 56) \cdot (73n \log n + 10^6)$

O Notation Defines Sets

$6n$ is $O(n)$ and $O(3n)$ and $O(2^n)$.
In fact, it is *all* of these:

$\vdots$
$O(2^n)$
$O(n^3)$
$O(6n - 99)$
$O(3n)$
$O(n + 8)$
$O(n)$

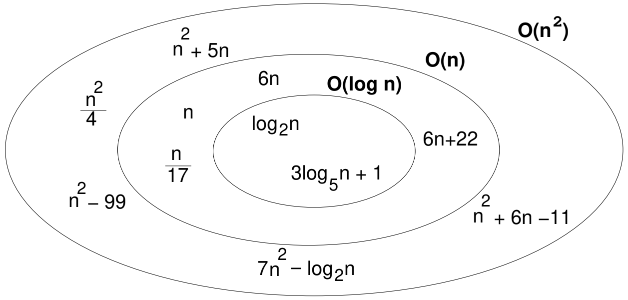
But it is not any of these:

$O(\log n)$
$O(\sqrt{n})$
$\vdots$

How can a function be both $O(n)$ and $O(3n)$, for example?

If some constant times n is an upper bound on the function (after some point B), then certainly some constant times $3n$ will be.

$O(n)$, e.g., defines a set containing all mathematical functions that are of that order.



Note that we always look for the smallest ("tightest") and simplest upper bound function that will satisfy the big-oh criteria.

E.g., for $6n$, $O(n)$ is the smallest upper bound instead of, say, $O(n^2)$,
and is also the simplest description of it instead of, say, $O(6n + 22)$.

Remember this

$O(1) \subset O(\log n) \subset O(n) \subset O(n \log n) \subset O(n^2) \subset O(n^3) \subset O(2^n) \subset O(3^n) \subset O(n!) \subset O(n^n)$

We can use this to determine which term in a mathematical function is the most dominant, and which other terms can be "cancelled".

Examples

- $O(5 \log n + n^2 + \frac{2n^3}{5})$

- $O(12n + n \log n)$

- $O(12n + n \log n + 2^n + n^2)$

Using O notation to Analyze the Running Time of Programs

Using very simple techniques, we can analyze code and know that the time to execute it is, for example,

$$O(n^2)$$

without having to know that the more detailed answer is, for example,

$$\frac{n^2 + 7 \log n}{3}$$

Example

```
static void silly (float num) {
    num = 0;
    num = (float) Math.sqrt(567.2);
    num = num / .000931f;
    System.out.println("num is " + num);
    System.out.println("Bye!");
}

       . . .

if (n > 1000) {
    System.out.println("That's big");
} else {
    System.out.println("That's not so big");
}
silly(n);
```

Analysis:

- This code involves no loops or recursion.
- Therefore it takes a constant amount of time, for some unknown constant.
- We call this “constant time” or $O(1)$.

Example

```
for (int i=100; i<=n; i++) {
    sum++;
}
```

Example

```
for (int i=1; i<=n/2; i++) {
    for (int j=1; j<=n*n; j++) {
        sum++;
    }
}
```

Analysis (working from the inside out):

- Each iteration of the inner loop takes $O(1)$ time.
- On every iteration of the outer loop, $O(n^2)$ iterations of the inner loop are performed.
- Thus each iteration of the outer loop takes $O(n^2)$ time.
- The outer loop is performed $\lfloor n/2 \rfloor$ times and $\lfloor n/2 \rfloor$ is $O(n)$.
- Therefore the loop takes $O(n^3)$ time.
- Therefore the program takes $O(n^3 + 1)$ time. (1 is for the initialization.)
- Thus the entire program takes $O(n^3)$ time.

Write your analyses in this style. Annotating the code is not sufficient.

Example

```
sum = 0;
for (int i=1; i<=n/2; i++) {
    sum++;
}
for (int j=1; j<=n*n; j++) {
    sum++;
}
```

Example

```
if (n % 2 == 0)
    for (int j=1; j<=n*n; j++) {
        sum++;
    }
} else {
    for (int k=5; k<=n+1; k++) {
        sum += k;
    }
}
```


Example

```
static boolean isSorted (int[] List) {
    // Details omitted.
    // Assume isSorted is implemented efficiently.
}

int[] myList = new int[size];
sum = 0;
if (isSorted(myList)) {
    for (int j=1; j<=n*n; j++) {
        sum++;
    }
} else {
    for (int k=5; k<=n+1; k++) {
        sum += k;
    }
}
```

Example

```
static void blah (int n) {
    int sum = 0;
    for (int i=1; i<=n/2; i++) {
        for (int j=1; j<=n*n; j++) {
            sum++;
        }
    }
    System.out.println (sum);
}
```

```
blah (p)                      blah (j*j)
```


Example

```
sum = 0;
while (num > 1) {
    num = num / 2;
    sum++;
}
```

Example

```
for (int k=1; k<=5000; k++) {
    if (a[k] % 2 == 0) {
        even++;
    } else {
        odd++;
    }
}
```


Example

```
for (int i=1; i<=n; i++) {
    for (int j=1; j<=m; j++) {
        sum++;
    }
}
```

Example

```
sum = 0;
for (int i=1; i<=n; i++) {
    for (int j=1; j<=i; j++) {
        sum++;
    }
}
```

Analysis:

- Each iteration of the inner loop takes $O(1)$ time.
- **On the ith iteration of the outer loop, $i \leq n$ iterations of the inner loop are performed.**
- Thus each iteration of the outer loop takes $O(n)$ time.
- The outer loop is performed n times.
- Thus the entire program takes $O(n^2)$ time.

Overestimating

$O(n^2)$ seems to be an overestimate.

The first time we do the inner loop, it iterates only once. On subsequent visits to the inner loop, it iterates more and and more times, until finally it does n iterations.

The total number of iterations of the inner loop is $1 + 2 + \cdots + n = n(n+1)/2$.

However, $n(n+1)/2$ is $O(n^2)$, so we get the same final answer, in terms of big-oh.

In some cases, the extra precision in the answer $n(n+1)/2$ may be important.

Summary of Rules

Type of code	Technique
no loops or recursion:	$O(1)$
loop:	multiply
sequence:	add
selection:	take the maximum
method call:	apply method's time to size of arguments