Faculty of Arts & Science / Physical Sciences Division
University of Toronto
CSC148/A58—Introduction to Computer Science, Fall 2000

Assignment 4: Proofs – Solution

1. Prove by induction that for all $n \geq 1$,

$$(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \cdots + (2 \times n - 1) = n^2.$$

Let $S(k)$ be the statement:

$$(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \cdots + (2 \times k - 1) = k^2.$$

We wish to prove that for all $n \geq 1$, $S(n)$ is true, and we do that by induction on n.

Base Case: $k = 1$. Then $S(1)$ is the statement: $(2 \times 1 - 1) = 1^2$ which is trivially true.

Induction Hypothesis: Let $k \geq 1$, and assume that $S(k)$ is true.

Inductive Step: We must prove that $S(k+1)$ is true. $S(k+1)$ is the statement:

$$(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \cdots + (2 \times (k+1) - 1) = (k+1)^2$$

Now,

$$(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \cdots + (2 \times (k+1) - 1) =$$
$$(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \cdots + (2 \times k - 1) + (2 \times (k+1) - 1).$$

By the induction hypothesis,

$$(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \cdots + (2 \times k - 1) = k^2$$

Therefore,

$$(2\times1-1)+(2\times2-1)+(2\times3-1)+\cdots+(2\times(k+1)-1) = k^2+(2\times(k+1)-1) = k^2+(2k+1) = (k+1)^2.$$

So, $S(k+1)$ is true.

By induction, we can conclude that $S(k)$ is true for all $k \geq 1$.

2. Consider the assertion:

"The number $n^2 + 5n + 1$ is even for all $n \geq 1$."

(a) Below is the outline of the induction step for a proof of this assertion. Complete it.

Let k be any integer ≥ 1.

Induction Hypothesis: Assume $k^2 + 5k + 1$ is even.

Induction Step: Prove that $(k+1)^2 + 5(k+1) + 1$ must also be even.

Body of the induction step missing.

(b) The assertion above is actually wrong. In fact, $n^2 + 5n + 1$ is *odd* for all $n \geq 1$. Explain how we can do the proof in (a) for an assertion that is wrong.

(c) Use mathematical induction to show that $n^2 + 5n + 1$ is odd for all $n \geq 1$.

(a) Let $S(k)$ be the statement "$k^2 + 5k + 1$ is even".

Induction Hypothesis: Let $k \geq 1$, and assume that $S(k)$ is true.

Inductive Step: We must prove that $S(k+1)$ is true. $S(k+1)$ is the statement:

$(k+1)^2 + 5(k+1) + 1$ is even.

Now,

$(k+1)^2 + 5(k+1) + 1 = k^2 + 7k + 7 = (k^2 + 5k + 1) + (2k + 6)$

By induction hypothesis, $(k^2 + 5k + 1)$ is even. Furthermore, 2 times any number is even.

Therefore,

$(k+1)^2 + 5(k+1) + 1 =$ even number + even number = even number.

So, $S(k+1)$ is true.

Note that we have not proven that the assertion is true for all $k \geq 1$ or even that it is true *for any single* k. All we have proven is that, for all $k \geq 1$, if $S(k)$ is true, so is $S(k+1)$. These things are not inconsistent. Since $S(k)$ is not true, neither is $S(k+1)$.

(b) **Base Case:** $k = 1$. Then $S(1)$ is the statement: $1^2 + 5 \times 1 + 1 = 7$ is even, which is obviously false. Since the base case is wrong, so is the assertion.

(c) Let $S(k)$ be the statement "$k^2 + 5k + 1$ is odd".

We wish to prove that for all $n \geq 1$, $S(n)$ is true, and we do that by induction on n.

Base Case: $k = 1$. Then $S(1)$ is the statement: $1^2 + 5 \times 1 + 1 = 7$ is odd, which is trivially true.

Induction Hypothesis: Let $k \geq 1$, and assume that $S(k)$ is true.

Inductive Step: We must prove that $S(k+1)$ is true. $S(k+1)$ is the statement:

$(k+1)^2 + 5(k+1) + 1$ is odd.

Now,

$(k+1)^2 + 5(k+1) + 1 = k^2 + 7k + 7 = (k^2 + 5k + 1) + (2k + 6)$

By the induction hypothesis, $(k^2 + 5k + 1)$ is odd

Therefore,

$(k+1)^2 + 5(k+1) + 1 =$ odd number + even number = odd number.

So, $S(k+1)$ is true.

By induction, we can conclude that $S(k)$ is true for all $k \geq 1$.

3. Consider attending a party where everyone shakes hands with everyone else. For example, at a three-person party attended by Sara, Tom and Bill, there will be 3 handshakes:

   Sara with Tom
   Sara with Bill
   Bill with Tom

   Use mathematical induction to prove that the total number of handshakes made in a party attended by n people ($n \geq 2$) is $n(n-1)/2$.

Let $S(k)$ be the statement:
"k people shake hands $k(k-1)/2$ times"

We wish to prove that for all $n \geq 2$, $S(n)$ is true, and we do that by induction on n.

Base Case: $k = 2$. Then $S(2)$ is the statement: "two people make $2(2-1)/2 = 1$ handshakes" which is trivially true.

Induction Hypothesis: Let $k \geq 2$, and assume that $S(k)$ is true.

Inductive Step: We must prove that $S(k+1)$ is true. $S(k+1)$ is the statement:

"$k+1$ people handshake $(k+1)k/2$ times"

Now, the number of handshakes between $k+1$ people is the number of handshakes between k people plus the number that the extra, $k+1$st person needs to make with everyone in the party. Since there are k other people in the party, the $k+1$st person needs to shake hands k times.

By the induction hypothesis,

the number of handshakes between k people is $k(k-1)/2$

Therefore,

the number of handshakes between $k+1$ people is $k(k-1)/2 + k = (k^2 - k + 2k)/2 = k(k+1)/2$. So, $S(k+1)$ is true.

By induction, we can conclude that $S(k)$ is true for all $k \geq 2$.

4. Consider the function F, defined by the following recurrence relation:

   $F_1 = 1$

   $F_2 = 1$

   $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$.

   So, $F_3 = F_1 + F_2 = 1 + 1 = 2$, etc. The values of F are called the *Fibonacci numbers*, after the Italian mathematician Leonardo Fibonacci, who first studied their properties in the year 1202.

   Prove that the sequence $F_3, F_6, F_9, \ldots$ consists of even numbers only.

Let $S(k)$ be the statement: "F_{3k} is even"

We wish to prove that for all $n \geq 1$, $S(n)$ is true, and we do that by induction on n.

Base Case: $k = 1$. Then $S(1)$ is the statement: "F_3 is even". $F_3 = F_2 + F_1 = 1 + 1 = 2$ which is even. Thus, $S(1)$ is true.

Induction Hypothesis: Let $k \geq 1$, and assume that $S(k)$ is true.

Inductive Step: We must prove that $S(k+1)$ is true. $S(k+1)$ is the statement:

"$F_{3(k+1)}$ is even"

Now

$F_{3(k+1)} = F_{3k+2} + F_{3k+1} = 2F_{3k+1} + F_{3k}$

By the induction hypothesis, F_3 is even. Furthermore, 2 times anything is even.

Therefore,

$F_{3(k+1)} =$ even number + even number = even number.

So, $S(k+1)$ is true.

By induction, we can conclude that $S(k)$ is true for all $k \geq 1$.

5. An advertisement for a tennis magazine says:

   (a) "If I am not playing tennis, I am watching tennis".

   (b) "If I am not watching tennis, I am reading about tennis".

   Presumably, each person can do only one thing at a time, so to this list we add

   (c) Noone can read about tennis and play tennis at the same time.

   (d) Noone can watch tennis and read about tennis at the same time.

   (e) Noone can watch tennis and play tennis at the same time

   Prove that the author of the ad is not reading about tennis.

Assume that the person is reading about tennis and do proof by contradiction.

If the person is reading about tennis, then
 by (5c), he/she is not playing tennis, and,
 by (5d), not watching tennis either.

If the person is not playing tennis, then
 by (5b), he/she is watching tennis.

So, the person is watching tennis and not watching tennis at the same time, leading to a contradiction.

Thus, the person is not reading about tennis.

6. The princess had two caskets, one gold and one silver. Into one she placed her portrait and into the other – a dagger. On the gold casket she wrote the inscription: "The portrait is not here". On the silver casket she wrote: "Exactly one of these inscriptions is true". She explained to the suitor that each inscription is either true or false (not both), but on the basis of the inscriptions he must choose the casket. If he chooses the one with the portrait, he can marry her; if he chooses the one with the dagger, he must kill himself.

   Prove that the portrait is in the golden casket.

Let's start by figuring out which inscription is true.

Assume that the second inscription is true.
 Then, the first one is false and the portrait is in the golden casket.

Assume that the second inscription is false.
 Then we have two cases:
 (a) Both inscriptions are false; in particular, the first one,
 and thus the portrait is in the golden casket.

 (b) Both inscriptions are true,
 which contradicts the second inscription.

Thus, by cases we can conclude that the portrait is in the golden casket.