CLRS 34-1 (a)

Definition 0.1. Let $G = (V, E)$ be an undirected graph. A subset $S \subseteq V$ is an independent set in G if there are no edges between vertices in S.

Independent Set Problem
Input: G - undirected graph.
Output: S - largest independent set in G.

Independent Set Problem, Decision Version
Input: G - undirected graph, $k \in \mathbb{N}$
Output: 1, if there exists an independent set of size k in G; 0, otherwise.

$L_{\text{IS}} = \{\langle G, k\rangle \mid G \text{ has independent set of size } k\}$.

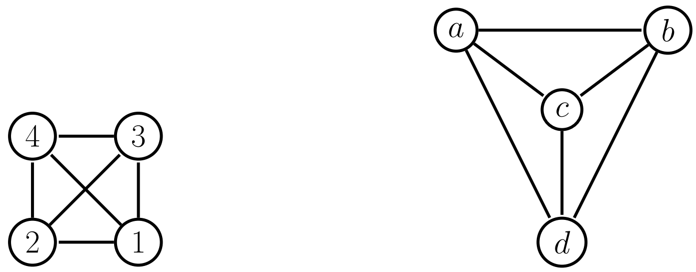
Claim 0.1. *L_{IS} is NP-complete*

Proof. (1) $L_{\text{IS}} \in \text{NP}$ — certificate is a set $S \subseteq V(G)$. The verifier checks that size of S is k and that each edge of G has at most one endpoint in S. Clearly, the certificate is of polynomial size in G and the verifier runs in polynomial time.
(2) $L_{\text{CLIQUE}} \leq_p L_{\text{IS}}$. Given graph $G = (V, E)$ and a number k, we need to construct $G' = (V', E')$ and k' in polytime such that G has a clique of size k if and only if G' has an independent set of size k. Construction is very simple, take the complement graph, i.e., $V' = V, E' = \binom{V}{2} \setminus E$ and $k' = k$. Note that set S is a clique in G if and only if S is an independent set in G'. Clearly, G' and k' can be computed in polytime. $\square$

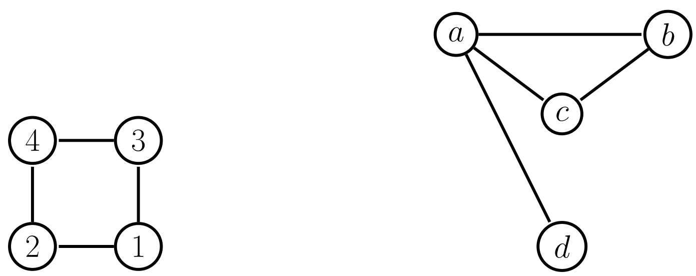
CLRS 34.5-1

Definition 0.2. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_{)}$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$. The bijection f is called isomorphism between G_1 and G_2.

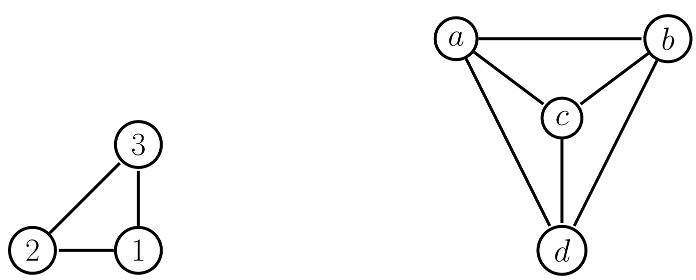
Example:
The following graphs are isomorphic by an isomorphism $f(1) = a, f(2) = b, f(3) = c, f(4) = d$.



The following two graphs are not isomorphic, because the second graph has a vertex of degree 1 and the first graph has only vertices of degree 2 and isomorphisms preserve degrees.



The following two graphs are not isomorphic because they do not have the same number of vertices.



Subgraph Isomorphism Problem
Input: G_1, G_2 — two undirected subgraphs
Ouput: 1, if G_1 is isomorphic to a **subgraph of** G_2; 0, otherwise

$L_{\text{SUB-ISO}} = \{\langle G_1, G_2 \rangle \mid G_1 \text{ is isomorphic to a subgraph of } G_2\}$.

Claim 0.2. *$L_{\text{SUB-ISO}}$ is NP-complete.*

Proof. (1) $L_{\text{SUB-ISO}} \in \text{NP}$ — the certificate is a subgraph $\widetilde{G} \subseteq G_2$ of G_2 and an isomorphism f between G_1 and $\widetilde{G}$. Clearly, the size of the certificate is polynomial in the size of the input. The verifier can check in polynomial time that $\widetilde{G}$ is a valid subgraph of G_2, and that f is a valid isomorphism. It simply verifies that f is a bijection — onto and one-to-one, and that f preserves the edges.
(2) $L_{\text{CLIQUE}} \leq_p L_{\text{SUB-ISO}}$. Given an instance of the clique problem, $\langle G, k \rangle$ we define G_1 to be a complete graph on k nodes, and let $G_2 = G$. Observe that G_1 is isomorphic to a subgraph of G_2 precisely when the corresponding sugraph is a k-clique. Thus, we have a valid reduction. Clearly, G_1 and G_2 can be computed in polynomial time. $\square$

Extra problems if you have time:

- CLRS 34.5-8.
- Prove that exact 4-SAT is NP-complete (reduce from exact 3-SAT).
- CLRS 34.4-6.