7.8 - Minimize Maximum Absolute Difference

Question (General)

You are given n points in a 2-D plane, $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$. Find a line $ax + by = c$ such that

$$\max_{i \in 1,2,\ldots n} |ax_i + by_i - c|$$

is minimized. Formulate this as a linear programming problem.

Solution (General)

To formulate this question as a linear program, we need to deal with the nonlinear max function. It is really difficult to find a linear system that is equivalent to $g = \max_{i \in 1,2,\ldots n} |ax_i + by_i - c|$, but it is possible to find some linear constraints that are equivalent to $g \geq \max_{i \in 1,2,\ldots n} |ax_i + by_i - c|$.

In fact, (1) and (2) below are equivalent.

$$g \geq \max_{i \in 1,2,\ldots n} |ax_i + by_i - c| \tag{1}$$

$$g \geq |ax_i + by_i + c| \text{ for each } i \in 1, 2, \ldots n \tag{2}$$

Since we know $g \geq |q|$ is equivalent to $g \geq q$ and $g \leq -q$, we can formulate the restraint $g \geq \max_{i \in 1,2,\ldots n} |ax_i + by_i - c|$ linearly as the following:

$$\begin{cases} g \geq ax_i + by_i - c \\ g \leq -ax_i - by_i + c \end{cases} \quad \text{for each } i \in 1, 2, \ldots n \tag{3}$$

Now, given the restraints in (3), if a, b and c are fixed as constant, minimizing g using linear programming algorithm will yield the maximum absolute difference, which is $\max_{i \in 1,2,\ldots n} |ax_i + by_i - c|$. If we treat a, b and c as free variables, minimizing g will yield the minimum value of maximum absolute difference which is as same as the question above.

To formulate the question above formally:

$$\begin{aligned} \text{minimize } & g \\ \text{subject to } & g \geq ax_i + by_i - c \text{ for each } i \in 1, 2, \ldots n \\ & g \leq -ax_i - by_i + c \text{ for each } i \in 1, 2, \ldots n \end{aligned}$$

g, a, b, c are all the free variables and x_i, y_i are the constants

The solution can end here since the question only asks for a linear program. For actually calculating the solution, we need to convert into a slack form. To accomplish that, we actually need some tricks (I summarize some of them at the end of the document).

We introduce slack variables $(p_i, q_i, a_1, a_2, b_1, b_2, c_1, c_2)$ as follows:

$$\begin{cases} p_i = g - ax_i - by_i + c \text{ for each } i \in 1, 2, \ldots n \\ q_i = -g - ax_i - by_i + c \text{ for each } i \in 1, 2, \ldots n \\ a_1 - a_2 = a \\ b_1 - b_2 = b \\ c_1 - c_2 = c \end{cases} \tag{4}$$

The slack form can be obtained as the following:

$$\begin{aligned} \text{maximize} \quad & -g \\ \text{subject to } p_i &= g - (a_1 - a_2)x_i - (b_1 - b_2)y_i + (c_1 - c_2) \text{ for each } i \in 1, 2, \dots n \\ q_i &= -g - (a_1 - a_2)x_i - (b_1 - b_2)y_i + (c_1 - c_2) \text{ for each } i \in 1, 2, \dots n \\ & g, a_1, a_2, b_1, b_2, c_1, c_2 \geq 0 \text{ and } p_i, q_i \geq 0 \text{ for each } i \in 1, 2, \dots n \end{aligned}$$

After solving this LP, the optimal solution obtained $(\bar{g}, \bar{a_1}, \bar{a_2}, \bar{b_1}, \bar{b_2}, \bar{c_1}, \bar{c_2})$ will represent the optimal line $(\bar{a_1} - \bar{a_2})x + (\bar{b_1} - \bar{b_2})y + (\bar{c_1} - \bar{c_2})$ and the minimized maximum deviation from the n points will be $\bar{g}$.

7.3 - Cargo Plane

Question

There are three types of materials to be transported. You may choose to carry any amount of each, upto the maximum available limits given below.

- Material 1 has density 2 tons/cubic meter, maximum available amount 40 cubic meters, and revenue \$1,000 per cubic meter.
- Material 2 has density 1 ton/cubic meter, maximum available amount 30 cubic meters, and revenue \$1,200 per cubic meter.
- Material 3 has density 3 tons/cubic meter, maximum available amount 20 cubic meters, and revenue \$12,000 per cubic meter.

Besides the limit for each individual material, the total weight limit is 100 tons and the total volume limit is 60 cubic meters. Write a linear program that optimizes revenue within the constraints.

Solution

Let x_1 be the volume of Material 1 to be carried, x_2 be the volume of Material 2 to be carried, x_3 be the volume of Material 3 to be carried.

The linear program can be formulated as follows:

$$\begin{aligned} \text{maximize } & 1000x_1 + 1200x_2 + 12000x_3 \\ \text{subject to } & x_1 + x_2 + x_3 \leq 60 \\ & 2x_1 + x_2 + 3x_3 \leq 100 \\ & x_1 \leq 40 \\ & x_2 \leq 30 \\ & x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Converting to slack form:

$$\begin{cases} z = 1000x_1 + 1200x_2 + 12000x_3 \\ x_4 = 60 - x_1 - x_2 - x_3 \\ x_5 = 100 - 2x_1 - x_2 - 3x_3 \\ x_6 = 40 - x_1 \\ x_7 = 30 - x_2 \\ x_8 = 20 - x_3 \end{cases} \tag{5}$$

Choose x_3 as the entering variable and x_8 as the leaving variable. We obtain our new slack form:

$$\begin{cases} z = 240000 + 1000x_1 + 1200x_2 - 12000x_8 \\ x_4 = 40 - x_1 - x_2 + x_8 \\ x_5 = 40 - 2x_1 - x_2 + 3x_8 \\ x_6 = 40 - x_1 \\ x_7 = 30 - x_2 \\ x_3 = 20 - x_8 \end{cases} \tag{6}$$

Choose x_2 as the entering variable and x_7 as the leaving variable. We obtain our new slack form:

$$\begin{cases} z = 276000 + 1000x_1 - 1200x_7 - 12000x_8 \\ x_4 = 10 - x_1 + x_7 + x_8 \\ x_5 = 10 - 2x_1 + x_7 + 3x_8 \\ x_6 = 40 - x_1 \\ x_2 = 30 - x_7 \\ x_3 = 20 - x_8 \end{cases} \tag{7}$$

Choose x_1 as the entering variable and x_5 as the leaving variable. We obtain our final slack form:

$$\begin{cases} z = 28100 - 500x_5 - 700x_7 - 10500x_8 \\ x_4 = 5 + 0.5x_5 + 0.5x_7 - 0.5x_8 \\ x_1 = 5 - 0.5x_5 + 0.5x_7 + 1.5x_8 \\ x_6 = 35 + 0.5x_5 - 0.5x_7 - 1.5x_8 \\ x_2 = 30 - x_7 \\ x_3 = 20 - x_8 \end{cases} \tag{8}$$

Solve for x_1, x_2 and x_3 by making $x_5 = x_7 = x_8 = 0$. We get $x_1 = 5$, $x_2 = 30$ and $x_3 = 20$.

Therefore, the optimal solution is to carry 5 cubic meter Material 1, 30 cubic meter Material 2 and 20 cubic meter Material 3. The maximum revenue is \$28100.

Summary of Tricks for Converting to Slack Form

Consider a linear program:

$$\begin{aligned} \text{maximize } & x_1 + x_2 + x_3 \\ \text{subject to } & x_1 + x_2 + x_3 \leq 10 \\ & x_1 - x_2 - x_3 \geq 3 \\ & x_1 + 2x_2 - x_3 = 7 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \end{aligned}$$

We can examine them case by case.

Converting a "≤" Constraint into Slack Form

In general, we can introduce one slack variable y to formulate the constraint:

$$a^T x \leq b \Leftrightarrow y = b - a^T x \geq 0$$

where y is the slack variable we create

Converting a "≥" Constraint into Slack Form

Similar to the previous case, we have:

$$a^T x \geq b \Leftrightarrow y = a^T x - b \geq 0$$

where y is the slack variable we create

Converting a "=" Constraint into Slack Form

In this case, we can introduce two new slack variables. That is:

$$a^T x = b \Leftrightarrow \begin{cases} y_1 = b - a^T x \geq 0. \\ y_2 = a^T x - b \geq 0. \end{cases}$$

Dealing with Non-positive Variables

If a variable x is not restrained as $x \leq 0$ instead of $x \geq 0$, we can replace the variable x using a slack variable y:

$$x \leq 0 \Leftrightarrow y = -x \geq 0$$

Dealing with Free Variables

If a variable x itself is not restrained, we can replace the variable x using two slack variables y_1 and y_2:

$$x \text{ as a free variable} \Leftrightarrow y_1 - y_2 = x, \text{ where } y_1, y_2 \geq 0$$

Apply to the Example Above

By applying the techniques above, we introduce a new set of variables $y_1, y_2, \ldots, y_8$ such that the following equalities hold.

$$\begin{cases} y_1 = 10 - x_1 - x_2 - x_3 \ (\leq \text{case}) \\ y_2 = x_1 - x_2 - x_3 - 3 \ (\geq \text{case}) \\ y_3 = 7 - x_1 - 2x_2 + x_3 \ (= \text{case}) \\ y_4 = x_1 + 2x_2 - x_3 - 7 \ (= \text{case}) \\ x_1 = y_5 \\ x_2 = -y_6 \ (\text{Non-positive variable}) \\ x_3 = y_7 - y_8 \ (\text{Free variable}) \end{cases} \tag{9}$$

By rearranging and substituting the last three equations back into the first four equations, we can obtain the slack form below:

$$
\begin{aligned}
\text{maximize } & y_5 - y_6 + y_7 - y_8 \\
\text{subject to } & y_1 = 10 - y_5 + y_6 - y_7 + y_8 \\
& y_2 = y_5 + y_6 - y_7 + y_8 \\
& y_3 = 7 - y_5 + 2y_6 + y_7 - y_8 \\
& y_4 = -7 + y_5 - 2y_6 - y_7 + y_8 \\
& y_1, y_2, \ldots, y_8 \geq 0
\end{aligned}
$$