**Working backwards.** If we wish to understand human behavior we should compare it with animal behavior. Animals also "have problems" and "solve problems." Experimental psychology has made essential progress in the last decades in exploring the "problem-solving" activities of various animals. We cannot discuss here these investigations but we shall describe sketchily just one simple and instructive experiment and our description will serve as a sort of comment upon the method of analysis, or method of "working backwards." This method, by the way, is discussed also elsewhere in the present book, under the name of PAPPUS to whom we owe an important description of the method.

1. Let us try to find an answer to the following tricky question: *How can you bring up from the river exactly six quarts of water when you have only two containers, a four quart pail and a nine quart pail, to measure with?*

Let us visualize clearly the given tools we have to work with, the two containers. (*What is given?*) We imagine two cylindrical containers having equal bases whose altitudes are as 9 to 4, see Fig. 24. *If* along the lateral surface of each container there were a scale of equally spaced horizontal lines from which we could tell the height of the waterline, our problem would be easy. Yet there is no such scale and so we are still far from the solution.

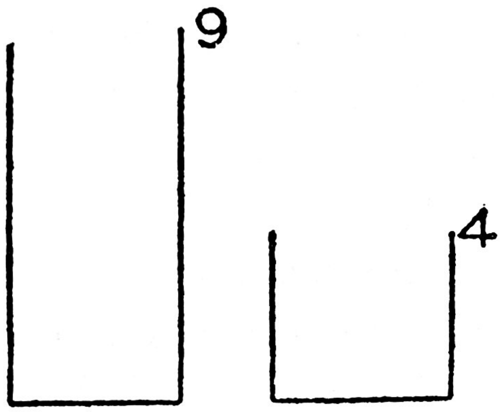


FIG. 24

We do not know yet how to measure exactly 6 quarts; but could we measure something else? (*If you cannot solve the proposed problem try to solve first some related problem. Could you derive something useful from the data?*) Let us do something, let us play around a little. We could fill the larger container to full capacity and empty so much as we can into the smaller container; then we could get 5 quarts. Could we also get 6 quarts? Here are again the two empty containers. We could also . . .

We are working now as most people do when confronted with this puzzle. We start with the two empty containers, we try this and that, we empty and fill, and when we do not succeed, we start again, trying something else. We are *working forwards*, from the given initial situation to the desired final situation, from the data to the unknown. We may succeed, after many trials, accidentally.

2. But exceptionally able people, or people who had the chance to learn in their mathematics classes something more than mere routine operations, do not spend too much time in such trials but turn around, and start working backwards.

What are we required to do? (*What is the unknown?*) Let us visualize the final situation we aim at as clearly as possible. Let us imagine that we have here, before us,

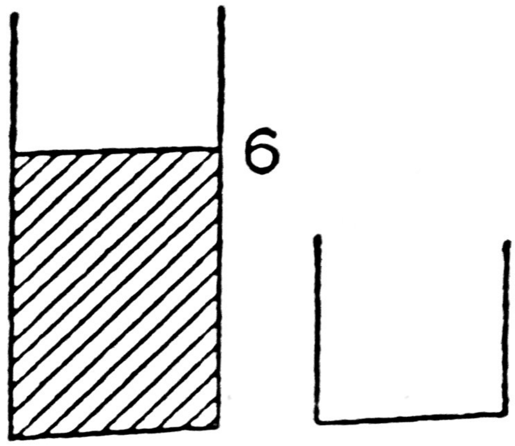


FIG. 25

the larger container with exactly 6 quarts in it and the smaller container empty as in Fig. 25. (Let us *start from what is required* and *assume what is sought as already found*, says Pappus.)

From what foregoing situation could we obtain the desired final situation shown in Fig. 25? (Let us *inquire from what antecedent the desired result could be derived*, says Pappus.) We could, of course, fill the larger container to full capacity, that is, to 9 quarts. But then we should be able to pour out exactly three quarts. To do that . . . we must have just one quart in the smaller container! That's the idea. See Fig. 26.

(The step that we have just completed is not easy at all. Few persons are able to take it without much foregoing hesitation. In fact, recognizing the significance of this step, we foresee an outline of the following solution.)

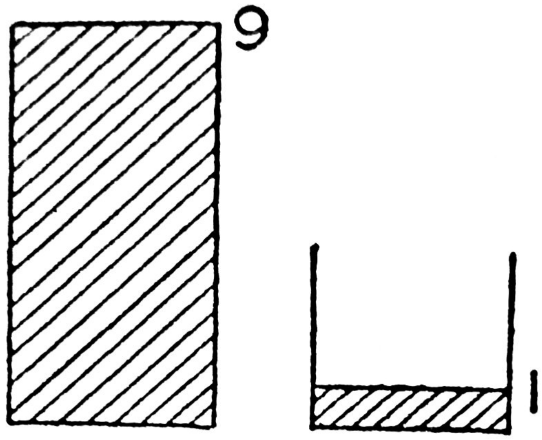


FIG. 26

But how can we reach the situation that we have just found and illustrated by Fig. 26? (Let us *inquire again what could be the antecedent of that antecedent.*) Since the amount of water in the river is, for our purpose, unlimited, the situation of Fig. 26 amounts to the same as the next one in Fig. 27

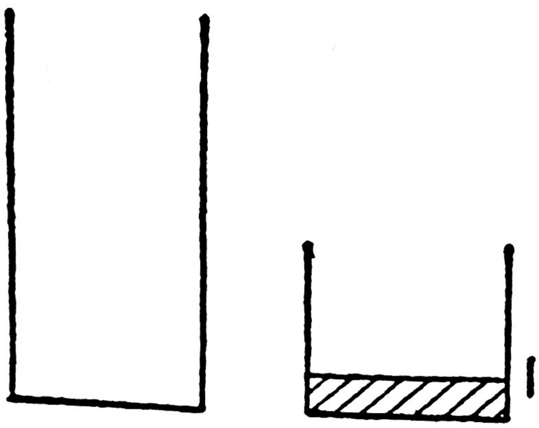


FIG. 27

or the following in Fig. 28.

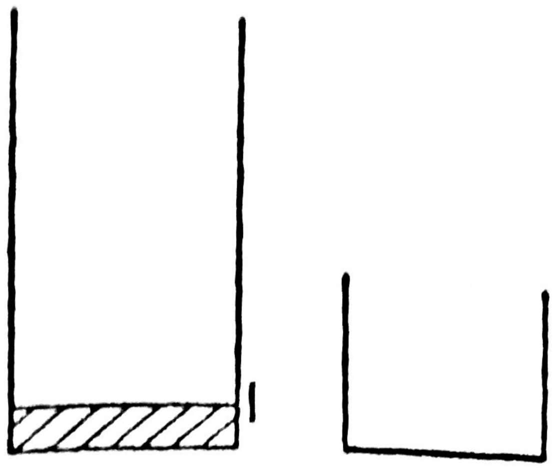

FIG. 28

It is easy to recognize that if any one of the situations in Figs. 26, 27, 28 is obtained, any other can be obtained just as well, but it is not so easy to hit upon Fig. 28, unless we *have seen it before,* encountered it accidentally in one of our initial trials. Playing around with the two containers, we may have done something similar and remember now, in the right moment, that the situation of Fig. 28 can arise as suggested by Fig. 29: We fill the large

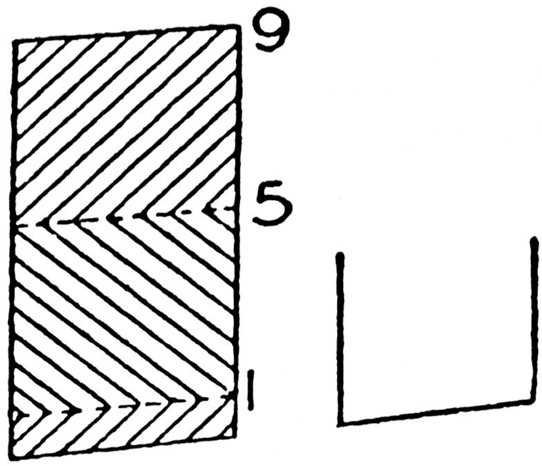


FIG. 29

container to full capacity, and pour from it four quarts into the smaller container and then into the river, twice in succession. We *came eventually upon something already known* (these are Pappus's words) and following the method of analysis, *working backwards,* we have discovered the appropriate sequence of operations.

It is true, we have discovered the appropriate sequence in retrogressive order but all that is left to do is to *reverse the process* and *start from the point which we reached last of all in the analysis* (as Pappus says). First, we perform the operations suggested by Fig. 29 and obtain Fig. 28; then we pass to Fig. 27, then to Fig. 26, and finally to Fig. 25. *Retracing our steps, we finally succeed in deriving what was required.*

3. Greek tradition attributed to Plato the discovery of the method of analysis. The tradition may not be quite reliable but, at any rate, if the method was not invented by Plato, some Greek scholar found it necessary to attribute its invention to a philosophical genius.

There is certainly something in the method that is not superficial. There is a certain psychological difficulty in turning around, in going away from the goal, in working backwards, in not following the direct path to the desired end. When we discover the sequence of appropriate operations, our mind has to proceed in an order which is exactly the reverse of the actual performance. There is some sort of psychological repugnance to this reverse order which may prevent a quite able student from understanding the method if it is not presented carefully.

Yet it does not take a genius to solve a concrete problem working backwards; anybody can do it with a little common sense. We concentrate upon the desired end, we visualize the final position in which we would like to be. From what foregoing position could we get there? It is natural to ask this question, and in so asking we work backwards. Quite primitive problems may lead naturally to working backwards; see PAPPUS, 4.

Working backwards is a common-sense procedure within the reach of everybody and we can hardly doubt that it was practiced by mathematicians and nonmathematicians before Plato. What some Greek scholar may have regarded as an achievement worthy of the genius of Plato is to state the procedure in general terms and to stamp it as an operation typically useful in solving mathematical and nonmathematical problems.

4. And now, we turn to the psychological experiment—if the transition from Plato to dogs, hens, and chimpanzees is not too abrupt. A fence forms three sides of a rectangle but leaves open the fourth side as shown in Fig. 30. We place a dog on one side of the fence, at the

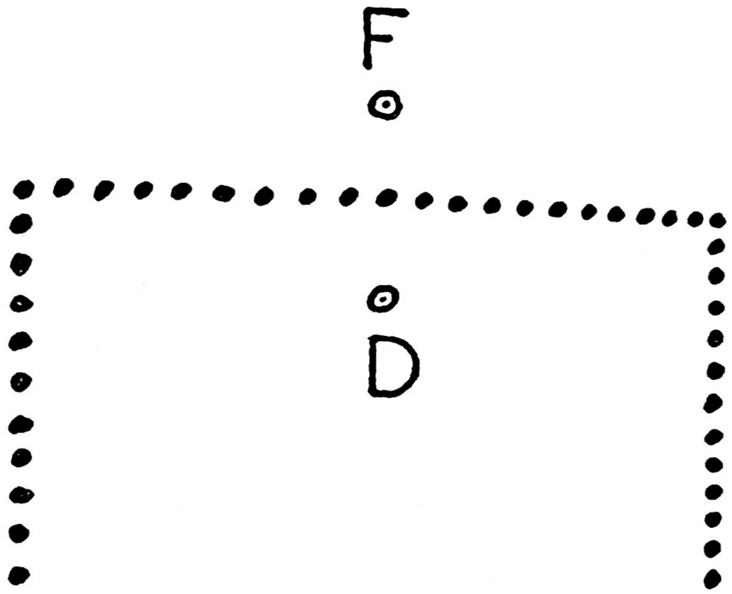


FIG. 30

point *D*, and some food on the other side, at the point *F*. The problem is fairly easy for the dog. He may first strike a posture as if to spring directly at the food but then he quickly turns about, dashes off around the end of the fence and, running without hesitation, reaches the food in a smooth curve. Sometimes, however, especially when the points *D* and *F* are close to each other, the solution is not so smooth; the dog may lose some time in barking, scratching, or jumping against the fence before he "conceives the bright idea" (as we would say) of going around.

It is interesting to compare the behavior of various animals put into the place of the dog. The problem is very easy for a chimpanzee or a four-year-old child (for whom a toy may be a more attractive lure than food). The problem, however, turns out to be surprisingly difficult for a hen who runs back and forth excitedly on her side of the fence and may spend considerable time before getting at the food if she gets there at all. But she may succeed, after much running, accidentally.

5. We should not build a big theory upon just one simple experiment which was only sketchily reported. Yet there can be no disadvantage in noticing obvious analogies provided that we are prepared to recheck and revalue them.

Going around an obstacle is what we do in solving any kind of problem; the experiment has a sort of symbolic value. The hen acted like people who solve their problem muddling through, trying again and again, and succeeding eventually by some lucky accident without much insight into the reasons for their success. The dog who scratched and jumped and barked before turning around solved his problem about as well as we did ours about the two containers. Imagining a scale that shows the waterline in our containers was a sort of almost useless scratching, showing only that what we seek lies deeper under the surface. We also tried to work forwards first, and came to the idea of turning round afterwards. The dog who, after brief inspection of the situation, turned round and dashed off gives, rightly or wrongly, the impression of superior insight.

No, we should not even blame the hen for her clumsiness. There is a certain difficulty in turning round, in going away from the goal, in proceeding without looking continually at the aim, in not following the direct path to the desired end. There is an obvious analogy between her difficulties and our difficulties.