Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

1. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with \( n \) marbles on each side, a total of \( \sum_{i=1}^{n} i \) marbles will be required.

   ![Triangle Diagram]

   \( T_1 = 1 \) \( T_2 = 3 \) \( T_3 = 6 \) \( T_4 = 10 \) \( T_5 = 15 \) \( T_6 = 21 \)

In lecture, we proved that \( \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \). For each \( n \in \mathbb{N} \), let \( T_n = \frac{n(n + 1)}{2} \); these numbers are usually called the **triangular numbers**. Use induction to prove that

\[
\forall n \in \mathbb{N}, \sum_{j=0}^{n} T_j = \frac{n(n + 1)(n + 2)}{6}
\]

**Solution**

Let us start by defining the predicate

\[
P(n) : \sum_{j=0}^{n} T_j = \frac{n(n + 1)(n + 2)}{6}
\]

We need to prove that \( \forall n \in \mathbb{N}, P(n) \).

**Proof.** **Base case:** let \( n = 0 \). We want to prove \( P(0) \). Then we can calculate:

\[
\sum_{j=0}^{n} T_j = \sum_{j=0}^{0} T_j = t_0 = \frac{0(0 + 1)}{2} = 0
\]
And also \( \frac{0(0 + 1)(0 + 2)}{6} = 0 \).

**Induction step:** Let \( k \in \mathbb{N} \) and assume \( P(k) \), i.e., that \( \sum_{j=0}^{k} T_j = k(k+1)(k+2)/6 \). We want to prove \( P(k+1) \), i.e., that \( \sum_{j=0}^{k} T_j = (k+1)(k+2)(k+3)/6 \).

We’ll calculate starting from the left side and show that it equals the right side.

\[
\sum_{j=0}^{k+1} T_j = \left( \sum_{j=0}^{k} T_j \right) + T_{k+1}
\]
\[
= \frac{k(k+1)(k+2)}{6} + T_{k+1} \quad \text{(by our assumption of } P(k) \text{)}
\]
\[
= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \quad \text{(by the definition of } T_{k+1} \text{)}
\]
\[
= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6}
\]
\[
= \frac{(k+1)(k+2)(k+3)}{6}
\]
2. **Induction (inequalities).** Consider the statement:

For every positive real number \( x \) and every natural number \( n \), \((1 + x)^n \geq 1 + nx\).

We can express the statement using the notation of predicate logic as:

\[
\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ (1 + x)^n \geq 1 + nx
\]

Notice that in this statement, there are two universally-quantified variables: \( n \) and \( x \). Prove the statement is true using the following approach:

(a) Use the standard proof structure to introduce \( x \).
(b) When proving the \(( \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx)\), do induction on \( n \).

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**Solution**

*Proof.* Let \( x \in \mathbb{R}^+ \). We’ll prove that for all \( n \in \mathbb{N} \), \((1 + x)^n \geq 1 + nx\) by induction.

**Base case:** Let \( n = 0 \).

Then \((1 + x)^n = 1\) and \(1 + nx = 1\). So then \((1 + x)^n \geq 1 + nx\).

**Induction step:** Let \( k \in \mathbb{N} \), and assume that \((1 + x)^k \geq 1 + kx\). We want to prove that \((1 + x)^k \geq 1 + (k+1)x\).

We’ll start with the quantity on the left, and show that it’s \( \geq \) the quantity on the right.

\[
(1 + x)^{k+1} = (1 + x)^k(1 + x) \\
\geq (1 + kx)(1 + x) \quad \text{(by our assumption)} \\
= 1 + kx + x + kx^2 \\
\geq 1 + kx + x \quad \text{(since } kx^2 \geq 0) \\
= 1 + (k + 1)x
\]

\( \square \)

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\(^1\) Your predicate \( P(n) \) that you want to prove will also contain the variable \( x \) – that’s okay, since when we do the induction proof, \( x \) has already been defined.
3. **Changing the starting number.** Recall that you previously proved that \( \forall n \in \mathbb{N}, \ n \leq 2^n \) using induction.

(a) First, use trial and error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!

\[
\forall n \in \mathbb{N}, n \geq \underline{8} \Rightarrow 30n \leq 2^n
\]

**Solution**

\[
\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n.
\]

(b) Now, prove the completed statement using induction. Be careful about how you choose your base case!

**Solution**

*Proof. Base case:* let \( n = 8 \).
Then \( 30n = 240 \), and \( 2^n = 256 \). So \( 30n \leq 64 \).

*Induction step:* let \( k \in \mathbb{N} \). Assume that \( k \geq 8 \), and that \( 30k \leq 2^k \). We want to prove that \( 30(k+1) \leq 2^{k+1} \).

Since \( 8 \leq k \), we know that \( 256 \leq 2^k \) (raising 2 to the power of either side). Our assumption tells us that \( 30k \leq 2^k \). Adding these two inequalities yields:

\[
\begin{align*}
30k + 256 & \leq 2^k + 2^k \\
30k + 256 & \leq 2^{k+1} \\
30k + 30 & \leq 2^{k+1} \\
30(k + 1) & \leq 2^{k+1} \quad \text{(since 30 \leq 256)}
\end{align*}
\]