Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

1. Induction (summations). If marbles are arranged to form an equilateral triangle shape, with \( n \) marbles on each side, a total of \( \sum_{i=1}^{n} i \) marbles will be required.

![Triangle diagrams]

In lecture, we proved that \( \sum_{i=1}^{n} i = n(n+1)/2 \). For each \( n \in \mathbb{N} \), let \( T_n = n(n+1)/2 \); these numbers are usually called the **triangular numbers**. Use induction to prove that

\[
\forall n \in \mathbb{N}, \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}
\]
2. **Induction (inequalities).** Consider the statement:

For every positive real number \( x \) and every natural number \( n \), \( (1 + x)^n \geq (1 + nx) \).

We can express the statement using the notation of predicate logic as:

\[
\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ (1 + x)^n \geq 1 + nx
\]

Notice that in this statement, there are two universally-quantified variables: \( n \) and \( x \). Prove the statement is true using the following approach:

(a) Use the standard proof structure to introduce \( x \).
(b) When proving the \((\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx)\), do induction on \( n \). \[\footnote{Your predicate \( P(n) \) that you want to prove will also contain the variable \( x \) – that’s okay, since when we do the induction proof, \( x \) has already been defined.}\]
3. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N}, n \leq 2^n$ using induction.

   (a) First, use trial and error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!
   
   $\forall n \in \mathbb{N}, n \geq \underline{\text{___________}} \Rightarrow 30n \leq 2^n$

   (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!