**Learning Objectives**

By the end of this worksheet, you will:

- Prove and disprove statements about numbers and functions.
- Use mathematical definitions of predicates to simplify or expand formulas.
- Identify errors in an incorrect proof.

1. **A direct proof.** Recall that we say an integer $n$ is **odd** if and only if $\exists k \in \mathbb{Z}, n = 2k - 1$. Using the technique from lecture, prove the following statement:

   For every pair of odd integers, their product is odd.

   Be sure to translate the statement into predicate logic. You can use the predicate $\text{Odd}(n)$: “$n$ is odd” in your formula without expanding the definition, but you’ll need to use the definition in your proof.

   **Solution**

   **Translation:**

   $\forall m, n \in \mathbb{Z}, \text{Odd}(m) \land \text{Odd}(n) \Rightarrow \text{Odd}(mn)$.

   **Discussion.** Like the proof we saw in lecture, we’ll need to use the definition of odd to introduce new variables to write $m = 2k_1 - 1$ and $n = 2k_2 - 1$; the rest should be straightforward algebra.

   **Proof.** Let $m, n \in \mathbb{Z}$, and assume they are both odd. That is, we assume there exist $k_1, k_2 \in \mathbb{Z}$ such that $m = 2k_1 - 1$ and $n = 2k_2 - 1$. We need to prove that $mn$ is odd, i.e., there exists $k_3$ such that $mn = 2k_3 - 1$. Let $k_3 = 2k_1k_2 - k_1 - k_2 + 1$.

   Then we can calculate:

   $$2k_3 - 1 = 2(2k_1k_2 - k_1 - k_2 + 1) - 1$$
   $$= 4k_1k_2 - 2k_1 - 2k_2 + 1$$
   $$= (2k_1 - 1)(2k_2 - 1)$$
   $$= mn$$

2. **An incorrect proof.** Consider the following claim:

   For every even integer $m$ and odd integer $n$, $m^2 - n^2 = m + n$.

   (a) Using the predicates $\text{Even}(n)$ and $\text{Odd}(n)$ (which return whether an integer $n$ is even or odd, respectively), express the above statement using the notation of symbolic logic.

   **Solution**

   $\forall m, n \in \mathbb{Z}, \text{Even}(m) \land \text{Odd}(n) \Rightarrow m^2 - n^2 = m + n$.

   (b) The following argument was submitted as a proof of the statement:
Proof. Let \( m \) and \( n \) be arbitrary integers, and assume \( m \) is even and \( n \) is odd. By the definition of even, \( \exists k \in \mathbb{Z}, m = 2k \); by the definition of odd, \( \exists k \in \mathbb{Z}, n = 2k - 1 \). We can then perform the following algebraic manipulations:

\[
m^2 - n^2 = (2k)^2 - (2k - 1)^2 = 4k^2 - 4k^2 + 4k - 1 = 4k - 1 = 2k + (2k - 1) = m + n
\]

The given argument is not a correct proof. What is the flaw? \( ^2 \)

Solution

The author has assumed that \( m = 2k \) and that \( n = 2k - 1 \), using the same variable \( k \) to express both \( m \) and \( n \). In this way, the author has unwittingly assumed that \( m \) and \( n \) are consecutive integers. But the statement is about an arbitrary \( m \) and an arbitrary \( n \), and so it is wrong to assume that they are consecutive numbers. The author should have let \( m = 2k_0 \) and \( n = 2k_1 - 1 \), for some integers \( k_0 \) and \( k_1 \). And of course, \( k_0 \) is not necessarily equal to \( k_1 \)!

3. Comparing functions. Consider the following definition:\( ^3 \)

**Definition 1.** Let \( f,g : \mathbb{N} \to \mathbb{R}^\geq \). We say that \( g \) is dominated by \( f \) (or \( f \) dominates \( g \)) if and only if for every natural number \( n \), \( g(n) \leq f(n) \).

(a) Express this definition symbolically by showing how to define the following predicate:

\[
\text{Dom}(f,g) : \quad \quad \quad \quad \quad \quad \quad , \text{ where } f,g : \mathbb{N} \to \mathbb{R}^\geq.
\]

**Solution**

\[
\text{Dom}(f,g) : \forall n \in \mathbb{N}, g(n) \leq f(n).
\]

(b) Let \( f(n) = 3n \) and \( g(n) = n \). Prove that \( g \) is dominated by \( f \).

**Solution**

We want to prove the following statement:

\[
\forall n \in \mathbb{N}, n \leq 3n
\]

**Proof.** Let \( n \in \mathbb{N} \).
Then since \( 1 \leq 3 \), we can multiply both sides by \( n \) to get \( n \leq 3n \). \( \square \)

(c) Let \( f(n) = n^2 \) and \( g(n) = n + 165 \). Prove that \( g \) is not dominated by \( f \). Make sure to write the statement you’ll prove in predicate logic, in fully simplified form (negations moved all the way inside).

**Solution**

The statement we want to prove is the negation of \( \text{Dom}(f,g) \):

\[
\exists n \in \mathbb{N}, n + 165 > n^2
\]

We leave the proof as an exercise.

\( ^2 \)If you have time, you might want to consider whether the given statement is true or false, and write a correct proof or disproof.

\( ^3 \)We’ll use the symbol \( \mathbb{R}^\geq \) to denote the set of all nonnegative real numbers, i.e., \( \mathbb{R}^\geq = \{ x \mid x \in \mathbb{R} \land x \geq 0 \} \).
(d) Now let’s generalize the previous statement. Translate the following statement into symbolic logic (expanding the definition of \textit{Dom}) and then prove it!

For every positive real number \( x \), \( g(n) = n + x \) is \textit{not} dominated by \( f(n) = n^2 \).

**Solution**

Translation:

\[
\forall x \in \mathbb{R}, \ x > 0 \Rightarrow (\exists n \in \mathbb{N}, \ n + x > n^2)
\]

For a proof, remember that zero is a natural number!
4. More with floor. Recall that the floor of a number \( x \), denoted \( \lfloor x \rfloor \), is the maximum integer less than or equal to \( x \). We can always write \( x = \lfloor x \rfloor + \epsilon \), where \( 0 \leq \epsilon < 1 \).

Prove the following statement\(^4\)

\[ \forall x \in \mathbb{R}^{\geq 0}, \ x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2 \]

Hint: First, prove the following simpler statement, and use it in your proof: \( \forall x \in \mathbb{R}^{\geq 0}, \ x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x \).

**Solution**

**Proof.** Let \( x \in \mathbb{R}^{\geq 0} \), and assume that \( x \geq 4 \). As noted in the question, we let \( \epsilon \in \mathbb{R} \) be defined as \( x - \lfloor x \rfloor \), and so \( 0 \leq \epsilon < 1 \).

Then we can calculate:

\[
(\lfloor x \rfloor)^2 = (x - \epsilon)^2 \\
= x^2 - 2x\epsilon + \epsilon^2 \\
\text{(1)}
\]

We now want to show that \( 2x\epsilon < \frac{1}{2}x^2 \), which we can do using our assumption \( x \geq 4 \):

\[
\begin{align*}
4 & \leq x \\
4x & \leq x^2 \\
2x & \leq \frac{1}{2}x^2 \\
2x\epsilon & \leq \frac{1}{2}x^2 \\
(\text{since} \ \epsilon < 1)
\end{align*}
\]

So then we can use this inequality in equation (1) to get:

\[
(\lfloor x \rfloor)^2 = x^2 - 2x\epsilon + \epsilon^2 \\
\geq x^2 - \frac{1}{2}x^2 + \epsilon^2 \quad \text{(since} \ 2x\epsilon \leq \frac{1}{2}x^2) \\
= \frac{1}{2}x^2 + \epsilon^2 \\
\geq \frac{1}{2}x^2
\]

\[\square\]

\(^4\)For extra practice, try proving the following generalization of this statement: \( \forall k \in \mathbb{R}^{\geq 0}, \ k < 1 \Rightarrow (\exists x_0 \in \mathbb{R}^{\geq 0}, \ \forall x \in \mathbb{R}^{\geq 0}, \ x \geq x_0 \Rightarrow (\lfloor x \rfloor)^2 \geq kx^2) \).