Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements about numbers and functions.
- Use mathematical definitions of predicates to simplify or expand formulas.
- Identify errors in an incorrect proof.

1. A direct proof. Recall that we say an integer \( n \) is \textbf{odd} if and only if \( \exists k \in \mathbb{Z}, \; n = 2k - 1 \). Using the technique from lecture, prove the following statement:

   For every pair of odd integers, their product is odd.

Be sure to translate the statement into predicate logic. You can use the predicate \( \text{Odd}(n) \): “\( n \) is odd” in your formula without expanding the definition, but you’ll need to use the definition in your proof.

2. An incorrect proof. Consider the following claim:

   For every even integer \( m \) and odd integer \( n \), \( m^2 - n^2 = m + n \).

   (a) Using the predicates \( \text{Even}(n) \) and \( \text{Odd}(n) \) (which return whether an integer \( n \) is even or odd, respectively), express the above statement using the notation of symbolic logic.

   (b) The following argument was submitted as a proof of the statement:

   \textit{Proof.} Let \( m \) and \( n \) be arbitrary integers, and assume \( m \) is even and \( n \) is odd. By the definition of even, \( \exists k \in \mathbb{Z}, \; m = 2k \); by the definition of odd, \( \exists k \in \mathbb{Z}, \; n = 2k - 1 \). We can then perform the following algebraic manipulations:

   \[
   m^2 - n^2 = (2k)^2 - (2k - 1)^2 \\
   = 4k^2 - 4k^2 + 4k - 1 \\
   = 4k - 1 \\
   = 2k + (2k - 1) \\
   = m + n
   \]
The given argument is not a correct proof. What is the flaw?\[ \text{[Explanation here]} \]

3. **Comparing functions.** Consider the following definition:  

**Definition 1.** Let \( f, g : \mathbb{N} \to \mathbb{R}^{\geq 0} \). We say that \( g \) is **dominated by** \( f \) (or \( f \) **dominates** \( g \)) if and only if for every natural number \( n \), \( g(n) \leq f(n) \).

(a) Express this definition symbolically by showing how to define the following predicate:

\[
\text{Dom}(f, g) : \quad \text{__________________________}, \quad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}.
\]

(b) Let \( f(n) = 3n \) and \( g(n) = n \). Prove that \( g \) is dominated by \( f \).

(c) Let \( f(n) = n^2 \) and \( g(n) = n + 165 \). Prove that \( g \) is not dominated by \( f \). Make sure to write the statement you’ll prove in predicate logic, in fully simplified form (negations moved all the way inside).

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1. If you have time, you might want to consider whether the given statement is true or false, and write a correct proof or disproof.
2. We’ll use the symbol \( \mathbb{R}^{\geq 0} \) to denote the set of all nonnegative real numbers, i.e., \( \mathbb{R}^{\geq 0} = \{ x \mid x \in \mathbb{R} \land x \geq 0 \} \).
(d) Now let’s generalize the previous statement. Translate the following statement into symbolic logic (expanding the definition of Dom) and then prove it!

For every positive real number $x$, $g(n) = n + x$ is not dominated by $f(n) = n^2$.

4. More with floor. Recall that the floor of a number $x$, denoted $\lfloor x \rfloor$, is the maximum integer less than or equal to $x$. We can always write $x = \lfloor x \rfloor + \epsilon$, where $0 \leq \epsilon < 1$.

Prove the following statement:\[\forall x \in \mathbb{R}^\geq 0, \ x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2} x^2\]

**Hint:** First, prove the following simpler statement, and use it in your proof: $\forall x \in \mathbb{R}^\geq 0, \ x \geq 4 \Rightarrow \frac{1}{2} x^2 \geq 2x$.

\[^3\text{For extra practice, try proving the following generalization of this statement: } \forall k \in \mathbb{R}^\geq 0, \ k < 1 \Rightarrow \left( \exists x_0 \in \mathbb{R}^\geq 0, \ \forall x \in \mathbb{R}^\geq 0, \ x \geq x_0 \Rightarrow (\lfloor x \rfloor)^2 \geq kx^2 \right).\]