More with induction!

Review (induction structure)

Statement to prove: \( \forall n \in \mathbb{N}, P(n) \).

What we actually prove:

\[
P(0) \land (\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1))
\]

base case

induction step

Proof

Base case: let \( n = 0 \). Want to prove \( P(0) \).

Induction step: let \( k \in \mathbb{N} \), and assume \( P(k) \).

Want to prove \( P(k+1) \).

(Today, variations)

Ex. Prove that...
\[ \forall n \in \mathbb{N}, n \geq 3 \Rightarrow 2n+1 < 2^n \]

We'll prove:

\[ P(3) \land \left( \forall k \in \mathbb{N}, P(k) \land k \geq 3 \Rightarrow P(k+1) \right) \]

Proof

Base case: let \( n = 3 \). Want to prove \( 2n+1 < 2^n \).

We calculate:

\[ 2n+1 = 7, \]

\[ 2^n = 8. \]

So then \( 2n+1 < 2^n \).

Induction step: Let \( k \in \mathbb{N} \). Assume \( 2k+1 < 2^k \) and that \( k \geq 3 \).
2^{k+1} < 2^k \quad \text{and that } k \geq 3.

\[ \text{Want to prove: } 2(k+1)+1 < 2^{k+1}. \]

\[ \text{body: Start with } 2k+2+1 \]

\[ 2k+1 < 2^k. \]

Since \( k \geq 3, \quad 2^k \geq 2^3 = 8 > 2. \)

So then we can add these inequalities:

\[ 2k+1 + 2 < 2^k + 2 \]

\[ 2(k+1) + 1 < \left[ 2^{k+1} \right] \]

\[ 2^k > 2 \]

\[ 2+2^k = 2 \cdot 2^k = 2^{k+1} \]

\[ a < b \]

\[ 2a < 2b \]

**Ex.** Prove that
∀x, y ∈ ℤ, ∀n ∈ ℕ,
5 | x - y ⇒ 5 | x^n - y^n.

Proof. Let x, y ∈ ℤ. Want to prove:
∀n ∈ ℕ, 5 | x - y ⇒ 5 | x^n - y^n.

We'll prove this using induction.
Define the predicate

P(n): 5 | x - y ⇒ 5 | x^n - y^n.

Base case: let n = 0. Want to prove that
5 | x - y ⇒ 5 | x^0 - y^0.

[Complete this for homework]

Induction step. Let k ∈ ℕ. Assume P(k), i.e.,
5 | x - y ⇒ 5 | x^k - y^k.

Want to prove: P(k+1), i.e.
5 | x - y ⇒ 5 | x^{k+1} - y^{k+1}
want to prove: $5 | x - y$.

Assume $5 | x - y$. Want to prove $5 | x^{k+1} - y^{k+1}$.

**Body:** Since we assumed $5 | x - y$ and that $5 | x - y \Rightarrow 5 | x^k - y^k$, we conclude $5 | x^k - y^k$.

We observe that

\[ x^{k+1} - y^{k+1} = x^k(x - y) + y^k(y - y^{k+1}) = x^k(x - y) + y^k(y - y^k) \]

this is divided by 5 
this is divided by 5

[exercise: use worksheet "LinComb" statement to finish proof.]