Contradiction and Induction

Ex (Contradiction)
There are infinitely many primes.

Translation:

\[ \forall n \in \mathbb{N}, \exists p \in \mathbb{N}, \text{Prime}(p) \land p > n. \]

Proof
Let \( n_0 \in \mathbb{N} \). Let \( p = n_0 + 1 \).

Want to prove \( \text{Prime}(p) \) and \( p > n_0 \).

Proof by contradiction

Want to prove statement \( P \).

Proof: Assume \( \neg P \). (i.e., \( P \) is false.)
Use this to prove $Q \land \neg Q$, for some statement $Q$.

\[
[-P \Rightarrow Q \land \neg Q] \quad \text{must be False}
\]

Contradiction

$Q$ could be anything! That's what makes this technique hard to use.

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**Proof** Assume the negation, i.e., that there are finitely many primes.

Then there exists $k \in \mathbb{N}$ and set

\[
Pr = \{p_1, p_2, \ldots, p_k\}
\]

such that

\[
\forall n \in \mathbb{N}, \text{Prime}(n) \iff n \notin Pr.
\]

(set of all primes.)
1. Q is true (follows from assumption)
2. Prove Q is false:

Let \( p = \prod_{i=1}^{k} p_i \cdot (p_k + 1) \).

1. \( p \) is not divisible by \( p_1 \), or by \( p_2 \), ..., or by \( p_k \).
2. \( p \) must be divisible by a prime.

Exercise: use 1 and 2 to prove \( \neg Q \).

Induction

Ex: Prove that for all \( n \in \mathbb{N} \),
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]
Proof attempt

Let $n \in \mathbb{N}$. Want to prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

body: ??????? ?

Induction.

Want to prove: $\forall n \in \mathbb{N}, P(n)$.

0 1 2 3 4

1) Prove $P(0)$. [Base case]
2) Prove $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$. [Induction step]

Proof by induction
Want to prove 1) and 2) for the predicate

\[ P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

Base case proof: Let \( n = 0 \).

L.S.: \( \sum_{i=1}^{0} i = 0 \) (empty sum)

R.S.: \( \frac{0(0+1)}{2} = 0 \).

Induction step: Prove \( \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1) \)

Let \( k \in \mathbb{N} \). Assume \( P(k) \), i.e., that \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \). Want to prove \( P(k+1) \), i.e., that \( \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \).
Start with \( \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \)

\[
\sum_{i=1}^{k} i = \frac{k(k+1)}{2} + k + 1
\]

by our assumption

\[
= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\]

\[
= \frac{(k+1)(k+2)}{2}
\]