Chapter 5: Graphs and Trees

Definitions

A graph is a tuple of sets, $(V, E)$, where

1. $V$ is a set, where each element is called a vertex.
2. $E$ is a set of edges, where
an edge is a pair of vertices: 
\[ \{v_i, u_i\}, \{v_i, u_i\} \]
\[ E = \{ (v_1, u_1), (v_2, u_2), (v_3, u_3), \ldots \} \]
where all \( v_i \)'s and \( u_i \)'s are in \( V \).

**Notes:**
- order doesn't matter in an edge \( \{v_i, u_i\} \)
- \( v_i \neq u_i \) (no "loops")
- at most one edge between 2 vertices
- all edges are "the same"

**Theorem**

For all graphs \( G = (V, E) \), \(|E| \leq \frac{|V|(|V| - 1)}{2} \).

**Proof**

\[ (\ldots) \]
Let $G = (V, E)$ be an arbitrary graph.

Each edge consists of exactly 2 vertices.

So the max # of edges is the # of subsets of $V$ of size 2.

[From Tutorial 6, this number is $\binom{|V|}{2}$]

Q: given a graph, is it always possible to get from one vertex to another?

Def'n: $\text{Adj}(G, u, v)$
Let $G = (V, E)$ be a graph, and $u, v \in V$. We say $u$ and $v$ are adjacent, or are neighbours, when $(u, v) \in E$.

A path between $u$ and $v$ is

a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ where:

1. $v_0 = u$
2. $v_k = v$
3. $\forall i \in \{0, 1, \ldots, k-1\}$, $v_i$ and $v_{i+1}$ are adjacent
4. All the $v$'s are distinct

The length of a path is its number of edges. (In the above diagram, this is $k$).
(In the above diagram, this is ky).

We say $u$ and $v$ are connected when $F$ a path between them.

We say the graph $G$ is (fully) connected when $\forall w_1, w_2 \in V, w_1$ and $w_2$ are connected.

The limits of connectedness

Main question today is

$\forall n \in \mathbb{N}, \exists m \in \mathbb{N},$

$\forall G = (V,E), (|V| = n \land |E| \geq M) \Rightarrow G \text{ is connected.}$
Proof

Let \( n \in \mathbb{N} \). Let \( M = (n - 1)n + 1 \).

For this choice of \( M \), the hypothesis is always false!!!

So then the whole statement is true.

(Vacuous truth)

We proved on Monday that \( |E| \leq \frac{n(n-1)}{2} \)

so it can't be \( \geq n(n-1)+1 \)

The main statement:

\[
\forall n \in \mathbb{N}, \forall G = (V, E), \quad (|V| = n \land |E| \geq \frac{(n-1)(n-2)}{2} + 1) \Rightarrow \\
G \text{ is connected.}
\]
Assume n vertices, n-1 of them all connected
Then add 1 edge, and the graph is now connected.

Real proof  Cool induction

\[ P(n) : \forall G = (V, E), (|V| = n \land |E| \geq \frac{(n-1)(n-2)}{2} + 1) \]
\[ \Rightarrow G \text{ is connected}. \]

The original statement is \( \forall n \in \mathbb{N}, P(n) \).

Induction step:

similar to set proofs