UNIVERSITY OF TORONTO
Faculty of Arts and Science

Midterm Examination 1
CSC236H1S

February 4, 2014 (50 min.)

Examination Aids: a single-sided, handwritten aid sheet

Last Name:
First Name:

Section (Circle One): L0101 (Th 4-6) L0201 (Th 11-12, 1-2)

Please read the following guidelines carefully!

• Please write your name on the front and back of the exam. The latter is to help us return the exams.

• This examination has three questions. There are a total of 7 DOUBLE-SIDED pages, not including this title page.

• Extra space is given after the questions for your scratch work or for overflow solutions.

• Answer questions clearly and completely. Give formal proofs unless explicitly asked not to. You may use any claim/result from class, unless you are being asked to prove that claim/result, or explicitly told not to.

• Any question you leave blank or clearly cross out your work and write “I don’t know” is worth 10% of the marks.

Take a deep breath.

This is your chance to show us

How much you’ve learned.

We WANT to give you the credit

That you’ve earned.

A number does not define you.

Good luck!
1. (a) [5] Prove using induction that for all \( n \geq 1 \), every set of natural numbers of size \( n \) has a maximum element. (The maximum element of a set is the element that is greater than all other elements. You may assume that sets aren’t allowed to contain repeated elements.)

(b) [2] Prove or disprove: every set of natural numbers has a maximum element. *Be sure to explain your reasoning!*
2. Recall that a binary string is a string of 0’s and 1’s. Consider the following recursively defined set $S$ of binary strings.

- $0 \in S$
- If $w \in S$, then $0w \in S$
- If $w \in S$, then $w1w \in S$

For example, the strings 0, 00, and 010 are all in $S$.

(a) [5] Prove using structural induction that every string in $S$ starts and ends with a 0.

(b) [2] Explain why $S$ does not equal the set of all binary strings that start and end with a 0.
3. Consider the following recursively defined function.

\[ f(n) = \begin{cases} 
5, & \text{if } n = 0 \\
2f(n-1) - 1, & \text{if } n \geq 1 
\end{cases} \]

(a) [4] Use repeated substitution to find a closed form expression for \( f(n) \). You do not need to prove that your closed form is correct, although if you have the time it’s an excellent way to check your work.

**WARNING:** your closed form may not contain any summations. You may find the following identity helpful:

\[
\forall x \geq 1, y > 1, \quad \sum_{i=0}^{k-1} xy^i = x \cdot \frac{y^k - 1}{y - 1}
\]
(b) [2] Now generalize your answer to part (a). Let $a, b, c \geq 1$, and $b \neq 1$. Consider the following recursively defined function:

$$g(n) = \begin{cases} 
a, & \text{if } n = 0 \\ bg(n-1) - c, & \text{if } n > 0 \end{cases}$$

Give a closed form expression for $g(n)$ (again, no summations). While you can use repeated substitution to do this, you do not need to show much work. You can get your answer directly by generalizing your work from part (a).
**Bonus Question [3]**

Consider the following extension of the game we saw in class. Let $m, n \geq 1$. The game starts with $m$ piles, each containing $n$ objects. Two players take turns removing an arbitrary number of objects from one of the piles; once a pile is empty, no more objects can be taken from it. The game ends when all piles are empty, and the winner is the player who removed the last object(s).

We saw in class that when $n = 2$, the second player can always win, if she is smart. State who has the winning strategy when there are $m$ piles, and prove your statement for all $m$, using induction.

**Warning:** this is a difficult question, and will be marked harshly. Only attempt it if you have finished all of the other questions!
Use this page for rough work.
Name:

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<th>Grade</th>
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