There are two components of this problem set: a preliminary question designed to check your understanding of the basic topics covered this week, and a set of more challenging questions designed to make you think critically about the material and apply it in new contexts. Get in the habit of starting work early – the less time you give yourself, the most stressed you’ll find yourself each week!

Caution: you must submit two separate files in two separate locations on MarkUs, one for the Preliminary and one for the Challenge.

To avoid suspicions of plagiarism, clearly state any resources (people, print, electronic) outside of your group, the course notes, and the course staff, you consulted at the beginning of your assignment submission.

Preliminary: due January 14, 2014 8:00 pm

This question is an opportunity for you to check your understanding of the topics and practice writing formal solutions. This is a valuable learning opportunity – if you see that you’re at a loss, get help quickly!

Your goal should not be to get the right answer, but to convince the marker that you know what you’re doing. This question is marked on the following 3-point scale:

<table>
<thead>
<tr>
<th>3: You’ve mastered this topic</th>
<th>1: You don’t really know what you’re doing</th>
</tr>
</thead>
<tbody>
<tr>
<td>2: You’re almost there, but missing something</td>
<td>0: You didn’t submit/had absolutely no clue</td>
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This question must be completed INDIVIDUALLY.

Prove, by induction, that \( \forall n \in \mathbb{N}, \) the units digit of \( 4^n \) is either 1, 4, or 6.

Remember that 0 \( \in \mathbb{N}. \) Hint: this question is also a review of using cases in proofs.

Challenges: due January 18, 2014 noon

Answer each question completely, always justifying your claims and reasoning. Your solution will not just be graded on correctness, but on its clarity as well. Technically correct answers that are hard to understand will not receive full marks.

Mark values for each question are contained in the [square brackets].

You may work in groups of up to THREE to complete these questions.

Note: The LaTeX Bonus is in effect for this problem set. See the course webpage for details.

1. It is well-known that every set of size \( n \) \((n \in \mathbb{N}) \) has exactly \( 2^n \) subsets. We can classify these subsets based on their number of elements. For example, if \( S \) is a set of size \( n \), then \( S \) contains exactly one subset of size 0 (the empty set), and \( n \) subsets of size 1 (simply pick one of the \( n \) elements).

(a) [5] Discover through experimentation/computation a formula for the number of two-element subsets of a set of size \( n \). Then, prove your formula is correct for all \( n \geq 0 \), using simple induction.

(b) [5] Prove that for all \( n \geq 0 \), every set of size \( n \) has \( \frac{n(n-1)(n-2)}{6} \) three-element subsets. You should use part (a) in your proof.

Do NOT appeal to “binomial coefficients” \( \binom{n}{k} \) in your solution! This defeats the whole purpose of the question.

2. Suppose you have an unlimited supply of 3- and 8-cent coins. It may be somewhat surprising that there exists a number \( m \in \mathbb{N} \) such that for every \( n \geq m \), it is possible to get a combination of 3- and 8-cent coins whose value is exactly \( n \).

(a) [6] Find the smallest possible value of \( m \), and then prove that \( \forall n \geq m \), it is possible to make \( n \) cents using 3- and 8-cent coins by simple induction. Do not use complete induction; this will be on your next problem set.

Hint: to use a simple induction argument, you must show how to make a value of \( k + 1 \) from a set of coins whose value is \( k \). Suppose you had a set of coins totalling 24; how could you turn that 24 into a 25?
(b) [2] Show that your $m$ from part (a) really is the smallest value possible by justifying why it is impossible to get the value $m - 1$ from 3- and 8-cent coins.

(c) [2] Finally, explain why your induction proof in part (a) FAILS if you start at $m - 1$ instead of $m$ (again, for the value of $m$ you found).

**Programming Question(s): purely for your benefit**

Occasionally, we will suggest programming exercises for you to try to reinforce the concepts presented in lecture and on the problem set. These are not to be handed in, but we think they can be very beneficial to your own learning (not to mention valuable coding practice). Feel free to discuss these questions with your peers and the course staff throughout the term!

Write a recursive program that takes in a list (representing a set) and outputs...

- All 2-element subsets of the list
- All 3-element subsets of the list
- All subsets of the list

Then, write a recursive program that takes in a list and a natural number $k$, and outputs all $k$-element subsets of the list.