3-repetition code on a 3-extension of the BSC

Assume the input alphabet $\{0, 1\}$. For the purpose of error correction, repeat each symbol 3 times at encoding. This produces the code words $\{000, 111\}$.

At decoding, choose the code word that is *closest* to a valid codeword.

Received code word	Hamming distance	Corrected code word
000	0	000
001	1	000
010	1	000
100	1	000
011	1	111
101	1	111
110	1	111
111	0	111

For example, assume the BSC has the reliability P, and the input has the probability distribution $p_0 = 0.6, p_1 = 0.4$.

The encoder's probability distribution is:

Code word	Probability
000	0.6
111	0.4
001	0
010	0
011	0
100	0
101	0
110	0

At decoding, we have the conditional probabilities

$$\text{Prob}(b_{\text{received}}|a_{\text{sent}}) = P^{n-d_{ab}}(1-P)^{d_{ab}} = P^n \left(\frac{P}{1-P}\right)^{d_{ab}}$$

where d_{ab} is the Hamming distance between the sent code word a, and the received code word b.

The forward probabilities P are:

P	000	001	010	011	100	101	110	111
000	0.729	0.081	0.081	0.009	0.081	0.009	0.009	0.001
111	0.001	0.009	0.009	0.081	0.009	0.081	0.081	0.729

Multiply the first line in P by $p_0 = 0.6$ and the second line by $p_1 = 0.4$, to obtain the joint probabilities R:

R	000	001	010	011	100	101	110	111
000	0.4374	0.0486	0.0486	0.0054	0.0486	0.0054	0.0054	0.0006
111	0.0004	0.0036	0.0036	0.0324	0.0036	0.0324	0.0324	0.2916

From R we obtain the backward probabilities Q:

Q	000	001	010	011	100	101	110	111
000	0.9991	0.9310	0.9310	0.1429	0.9310	0.1429	0.1429	0.0021
111	0.0009	0.0690	0.0690	0.8571	0.0690	0.8571	0.8571	0.9979

Let's see if our correction scheme (i.e. choosing the closest correct code word) is effective.

Assume that "000" is received; the results can be taken directly from Q.

$$\text{Prob}(a = 000|b = 000) = 0.6\frac{0.729}{0.4378} = 0.9991$$

is bigger than

$$\text{Prob}(a = 111|b = 000) = 0.4\frac{0.001}{0.4378} = 0.0009$$

therefore, if 000 is received, conclude that 000 was sent: $\Delta(000) = 000 \leftrightarrow 0$.

Now, assume that "101" is received.

$$\text{Prob}(a = 000|b = 101) = 0.1429$$

is smaller than

$$\text{Prob}(a = 111|b = 101) = 0.8571$$

therefore, if 101 is received, conclude that 111 was sent: $\Delta(101) = 111 \leftrightarrow 1$.

However, if the channel reliability is not big enough, the correction scheme does not function properly. In the following examples, p_1 is small, and P is not big enough to account for the size of p_1:

$$p_1 P^3 < p_0(1-P)^3 \Rightarrow \Delta(111) = 000$$

$$p_1 P < p_0(1-P) \Rightarrow \Delta(011) = \Delta(101) = \Delta(110) = 000$$