UNIVERSITY OF TORONTO

Faculty of Arts and Science

term test #1 SOLUTIONS
CSC236

Date: Monday February 3, 2020
Duration: 50 minutes
Instructor(s): Colin Morris

Examination Aids: pencils, pens, erasers, drinks, snacks

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first and last names:

utorid:

student number:

Please read the following guidelines carefully!

- **Please write your name, utorid, and student number on the front of this exam.**
- This examination has **3** questions. There are a total of **7 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely.
- You will receive 20% of the marks for any question you leave blank or indicate "I cannot answer this question."

Take a deep breath.
This is your chance to show us
How much you've learned.

We **WANT** to give you the credit

Good luck!

1. **[5 marks]** ($\approx$ 10 minutes) The table below contains variations on the structure of an inductive proof. Fill in the last column to indicate the set of numbers $\subseteq \mathbb{N}$ for which we can conclude the predicate P holds, having proven the given base cases and inductive steps. If you think there are none, use the symbol $\varnothing$. The first row has been filled in as an example.

Basis	**Inductive step**	**Therefore $P(n)$ holds for...**
$P(0)$	$\forall n \in \mathbb{N}, P(n) \implies P(n+2)$	even numbers
$P(236)$	$\forall n \in \mathbb{N}, P(n) \implies P(n+1)$	$n \geq 236$
-	$\forall n \in \mathbb{N}, [\forall k \in \mathbb{N}, 0 < k < n \implies P(k)] \implies P(n)$	all n
-	$\forall n \in \mathbb{N}, [\forall k \in \mathbb{N}, k \leq n \implies P(k)] \implies P(n+1)$	$\varnothing$
$P(0) \wedge P(1)$	$\forall n \in \mathbb{N}, P(n) \implies P(2n+1)$	$n = 2^k - 1$ for $k \in \mathbb{N}$
$P(0)$	$\forall n \in \mathbb{N}, n > 0 \wedge P(n-1) \implies P(n)$	all n

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2. **[9 marks]** ($\approx$ 18 minutes) $\mathcal{F}$ is a set of strings representing a limited class of proposition formulas, using only implication, negation and a finite number of variables (or 'atoms'). In particular, we define $\mathcal{F}$ to be the smallest set of strings such that:

   (1) x, y, and z are elements of $\mathcal{F}$. (We'll refer to these as 'atoms'.)

   (2) If $f_1, f_2 \in \mathcal{F}$ then $(f_1 \Rightarrow f_2) \in \mathcal{F}$.

   (3) If $f \in \mathcal{F}$ then $\neg f \in \mathcal{F}$.

   Denote the number of atoms in string f by $A(f)$, and the number of parentheses by $B(f)$.
   For example, $f' = \neg((x \Rightarrow y) \Rightarrow \neg x)$ is an example of an element in $\mathcal{F}$ having $A(f') = 3$ and $B(f') = 4$ (note that we count duplicate atoms).
   Use structural induction to prove that $\forall f \in \mathcal{F}, B(f) \geq A(f) - 1$.

   Solution Define $P(f) : B(f) \geq A(f) - 1$.
   Basis: Let $f \in \{x, y, z\}$. Then $B(f) = 0$ and $A(f) = 1$, so $B(f) = 0 \geq 1 - 1 = A(f) - 1$, so $P(f)$.
   Inductive step (rule 2): Let $f_1, f_2 \in \mathcal{F}$ and assume $P(f_1) \wedge P(f_2)$. Let f be the formula $(f_1 \implies f_2)$. Then by definition, we have

$$B(f) = 2 + B(f_1) + B(f_2) \tag{1}$$
$$A(f) = A(f_1) + A(f_2) \tag{2}$$

   Applying the I.H. to (1), we get

$$\begin{aligned} B(f) &\geq 2 + (A(f_1) - 1) + (A(f_2) - 1) \\ &= A(f) \qquad \text{\# simplifying and substituting (2)} \\ &\geq A(f) - 1 \end{aligned}$$

   Thus $P(f)$.
   Inductive step (rule 3): Let $f_1 \in \mathcal{F}$ and assume $P(f_1)$. Let f be the formula $\neg f_1$. $A(f) = A(f_1)$ and $B(f) = B(f_1)$, so $P(f)$ follows immediately from our I.H.

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3. **[9 marks]** ($\approx$ 22 minutes) Consider a game played between two players, P1 and P2, with the following rules:

- The game starts with a box containing $n > 0$ chopsticks
- Until the game is won, P1 and P2 alternate making moves, with P1 making the first move
- A valid move is to remove either 1, 2, or 3 chopsticks from the box
- The player who causes the box to be empty (by removing the last chopstick(s)) wins

Use complete induction to prove that for all $n > 0$, if n is not a multiple of 4, then if the box starts with n chopsticks, P1 can win the game no matter what P2 does.
Suggestion: Before starting your proof, it may help to think about a few small values of n. What does P1 need to do to guarantee a win?

Solution Define $P(n) : (n \bmod 4 \neq 0) \implies$ starting from a box of n chopsticks, P1 can win the game.

I will use complete induction to prove $\forall n \in \mathbb{N}^+, P(n)$.

Let $n \in \mathbb{N}^+$. Assume $\forall k \in \mathbb{N}, 0 < k < n \implies P(k)$.

<u>Case 1: $0 < n < 4$</u>
Then P1 can immediately win the game by taking all n chopsticks, so $P(n)$.
<u>Case 2: $n \bmod 4 = 0$</u>
Then $P(n)$ is vacuously true.
<u>Case 3: $n \geq 4 \wedge n \bmod 4 \neq 0$</u>
In this case, I will show that P1 can win if they take $n \bmod 4$ chopsticks. This leaves the box with $4k$ chopsticks when it passes to P2, for some $k \in \mathbb{N}^+$. P2 must then remove 1, 2, or 3 chopsticks, leaving $c \in \{4k-1, 4k-2, 4k-3\}$ chopsticks when the box returns to P1.
Note that the following facts are true of all possible values of c:

- $0 < c < n$, meaning that $P(c)$ holds by our I.H.
- $c \bmod 4 \neq 0$

Therefore, by $P(c)$, P1 can win the game from this point, so $P(n)$ holds.

$P(n)$ holds in all cases. $\blacksquare$

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