UNIVERSITY OF TORONTO
Faculty of Arts and Science
term test #1 SOLUTIONS
CSC236
Date: Monday February 3, 2020
Duration: 50 minutes
Instructor(s): Colin Morris
Examination Aids: pencils, pens, erasers, drinks, snacks

first and last names:
utorid:
student number:

Please read the following guidelines carefully!

- Please write your name, utorid, and student number on the front of this exam.
- This examination has 3 questions. There are a total of 7 pages, DOUBLE-SIDED.
- Answer questions clearly and completely.
- You will receive 20% of the marks for any question you leave blank or indicate “I cannot answer this question.”

Take a deep breath.
This is your chance to show us
How much you’ve learned.

We WANT to give you the credit
Good luck!
1. [5 marks] (≈ 10 minutes) The table below contains variations on the structure of an inductive proof. Fill in the last column to indicate the set of numbers ⊆ N for which we can conclude the predicate P holds, having proven the given base cases and inductive steps. If you think there are none, use the symbol ∅. The first row has been filled in as an example.

<table>
<thead>
<tr>
<th>Basis</th>
<th>Inductive step</th>
<th>Therefore P(n) holds for...</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(0)</td>
<td>∀n ∈ N, P(n) =⇒ P(n + 2)</td>
<td>even numbers</td>
</tr>
<tr>
<td>P(236)</td>
<td>∀n ∈ N, P(n) =⇒ P(n + 1)</td>
<td>n ≥ 236</td>
</tr>
<tr>
<td>-</td>
<td>∀n ∈ N, [∀k ∈ N, 0 &lt; k &lt; n =⇒ P(k)] =⇒ P(n)</td>
<td>all n</td>
</tr>
<tr>
<td>-</td>
<td>∀n ∈ N, [∀k ∈ N, k ≤ n =⇒ P(k)] =⇒ P(n + 1)</td>
<td>∅</td>
</tr>
<tr>
<td>P(0) ∧ P(1)</td>
<td>∀n ∈ N, P(n) =⇒ P(2n + 1)</td>
<td>n = 2^k − 1 for k ∈ N</td>
</tr>
<tr>
<td>P(0)</td>
<td>∀n ∈ N, n &gt; 0 ∧ P(n − 1) =⇒ P(n)</td>
<td>all n</td>
</tr>
</tbody>
</table>
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2. [9 marks] \( \approx \) 18 minutes  
\( F \) is a set of strings representing a limited class of proposition formulas, using only implication, negation and a finite number of variables (or ‘atoms’). In particular, we define \( F \) to be the smallest set of strings such that:

1. \( x, y, \) and \( z \) are elements of \( F \). (We’ll refer to these as ‘atoms’.)
2. If \( f_1, f_2 \in F \) then \( (f_1 \Rightarrow f_2) \in F \).
3. If \( f \in F \) then \( \neg f \in F \).

Denote the number of atoms in string \( f \) by \( A(f) \), and the number of parentheses by \( B(f) \). For example, \( f' = \neg((x \Rightarrow y) \Rightarrow \neg x) \) is an example of an element in \( F \) having \( A(f') = 3 \) and \( B(f') = 4 \) (note that we count duplicate atoms).

Use structural induction to prove that \( \forall f \in F, B(f) \geq A(f) - 1 \).

Solution  
Define \( P(f) : B(f) \geq A(f) - 1 \).

Basis: Let \( f \in \{x, y, z\} \). Then \( B(f) = 0 \) and \( A(f) = 1 \), so \( B(f) = 0 \geq 1 - 1 = A(f) - 1 \), so \( P(f) \).

Inductive step (rule 2): Let \( f_1, f_2 \in F \) and assume \( P(f_1) \land P(f_2) \). Let \( f \) be the formula \( (f_1 \Rightarrow f_2) \). Then by definition, we have

\[
B(f) = 2 + B(f_1) + B(f_2) \quad \text{(1)} \\
A(f) = A(f_1) + A(f_2) \quad \text{(2)}
\]

Applying the I.H. to (1), we get

\[
B(f) \geq 2 + (A(f_1) - 1) + (A(f_2) - 1) \\
= A(f) \quad \# \text{simplifying and substituting (2)} \\
\geq A(f) - 1
\]

Thus \( P(f) \).

Inductive step (rule 3): Let \( f_1 \in F \) and assume \( P(f_1) \). Let \( f \) be the formula \( \neg f_1 \). \( A(f) = A(f_1) \) and \( B(f) = B(f_1) \), so \( P(f) \) follows immediately from our I.H.
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3. [9 marks] (≈ 22 minutes) Consider a game played between two players, P1 and P2, with the following rules:

- The game starts with a box containing \( n > 0 \) chopsticks
- Until the game is won, P1 and P2 alternate making moves, with P1 making the first move
- A valid move is to remove either 1, 2, or 3 chopsticks from the box
- The player who causes the box to be empty (by removing the last chopstick(s)) wins

Use complete induction to prove that for all \( n > 0 \), if \( n \) is not a multiple of 4, then if the box starts with \( n \) chopsticks, P1 can win the game no matter what P2 does.

**Suggestion:** Before starting your proof, it may help to think about a few small values of \( n \). What does P1 need to do to guarantee a win?

**Solution** Define \( P(n) : (n \text{ mod } 4 \neq 0) \implies \text{starting from a box of } n \text{ chopsticks, P1 can win the game.} \)

I will use complete induction to prove \( \forall n \in \mathbb{N}^+, P(n) \).

Let \( n \in \mathbb{N}^+ \). Assume \( \forall k \in \mathbb{N}, 0 < k < n \implies P(k) \).

Case 1: \( 0 < n < 4 \)
Then P1 can immediately win the game by taking all \( n \) chopsticks, so \( P(n) \).

Case 2: \( n \text{ mod } 4 = 0 \)
Then \( P(n) \) is vacuously true.

Case 3: \( n \geq 4 \land n \text{ mod } 4 \neq 0 \)
In this case, I will show that P1 can win if they take \( n \text{ mod } 4 \) chopsticks. This leaves the box with \( 4k \) chopsticks when it passes to P2, for some \( k \in \mathbb{N}^+ \). P2 must then remove 1, 2, or 3 chopsticks, leaving \( c \in \{4k-1, 4k-2, 4k-3\} \) chopsticks when the box returns to P1.

Note that the following facts are true of all possible values of \( c \):

- \( 0 < c < n \), meaning that \( P(c) \) holds by our I.H.
- \( c \text{ mod } 4 \neq 0 \)

Therefore, by \( P(c) \), P1 can win the game from this point, so \( P(n) \) holds.

\( P(n) \) holds in all cases. \( \blacksquare \)
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