CSC236 winter 2020, week 6: Recursive correctness

Recommended supplementary reading: Chapter 2 Vassos course notes

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Master Theorem proof sketch

A postscript to last week

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d) \xRightarrow{\text{Master Theorem}} T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_b n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

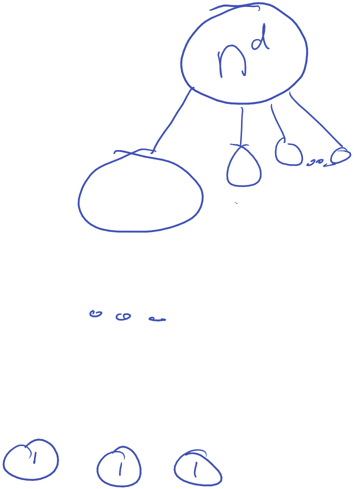
$$\begin{aligned} T(n) &= n^d + aT(n/b) \\ &= n^d + a((n/b)^d + aT(n/b^2)) \\ &= n^d + a((n/b)^d + a((n/b^2)^d + aT(n/b^3))) \\ &= n^d + a(n/b)^d + a^2(n/b^2)^d + a^3T(n/b^3) \\ &\dots \\ &= \sum_{i=0}^{\log_b n} a^i (n/b^i)^d = \sum a^i \frac{n^d}{b^{di}} = n^d \times \sum_{i=0}^{\log_b n} \frac{a^i}{b^{di}} \end{aligned}$$

last term

$$\frac{a^{\log_b n}}{b^{d \log_b n}}$$

Master Theorem proof sketch

$T(n) = n^d + aT(n/b)$



Input size		Work per node
n	a^0 nodes	n^d
n/b	a nodes	$(n/b)^d$
n/b^2	a^2 nodes	$(n/b^2)^d = n^d / b^{d2}$
...	...	

$1 = \frac{n}{n} = \frac{n}{b^{\log_b n}}$

$a^{\log_b n}$

Total work at leaf layer

$a^{\log_b n} = n^{\log_b a}$

Proving correctness

- Formally proving that my code will 'do the right thing' on *any* appropriate input
- Contrast with unit testing: verifies that my code does the right thing on certain specific inputs
  - "Beware of bugs in the above code; I have only proved it correct, not tried it." - Donald Knuth
- Do people do this in the real world? Yes! (Sometimes)
  - Think of code that runs on medical devices or airplanes

Example: factorial

```
1 def fact(n):
2   """Return n!
3   """
4   if n == 0:
5     return 1
6   return n * fact(n-1)
```

$P(n)$: fact(n) = n!

Basis fact(0) = 1 = 0!
↑ by lines 4-5
so P(0)

Let n ∈ ℕ, assume P(n)

n+1 > 0, so we reach l6

so fact(n+1) = (n+1) × fact(n)

= (n+1) × n!   # by IH

= (n+1)!

so P(n+1)

A more complex example: max

```
def max(A):
  if len(A) == 1:
    return A[0]
  max_tail = max(A[1:])
  if A[0] > max_tail:
    return A[0]
  else:
    return max_tail
```

P(n):

A more complex example: `max`

max([]) → IndexError

```python
def max(A):
  if len(A) == 1:
    return A[0]
  max_tail = max(A[1:])
  if A[0] > max_tail:
    return A[0]
  else:
    return max_tail
```

$P(n)$: For any list `A` of length `n`, `max(A)` returns the maximum element of that list.

- Is $P(0)$ true? ✗
- Is $P(2)$ true?

$\max(A) = x$ s.t. $\forall i \; A[i] \leq x$

→ A = [None, 1]

Correctness can only be defined relative to some *specifications*

- **Precondition**: For what inputs is the behaviour of my algorithm defined?
  - Note: my algorithm can do *anything* on other inputs (including crash)
- **Postcondition**: What *should* be true after my algorithm terminates?
  - Usually describes the return value. Or, for an 'in-place' algorithm, describes how the inputs have changed.
- 'My algorithm is correct' $\equiv$ Whenever the precondition holds, my algorithm terminates, and the postcondition is true.

(Often these will be given to you along with the algorithm. Sometimes you'll have to come up with reasonable ones to make your proof work.)

Specifications for `max`

```python
def max(A):
  if len(A) == 1:
    return A[0]
  max_tail = max(A[1:])
  if A[0] > max_tail:
    return A[0]
  else:
    return max_tail
```

Pre(A): $\text{len}(A) > 0$, and all elements of A are comparable

Post(A): $\text{max}(A) = x$, s.t. $\forall i \in \mathbb{N}\ \ i < \text{len}(A) \Rightarrow A[i] \leq x \ \wedge\ \exists j \in \mathbb{N}, x = A[j]$

$P(n)$: for any list `A`, $\texttt{len(A)} = n \wedge \text{Pre}(A) \implies \text{Post}(A)$[1]

[1] You don't need to define separate predicates for pre- and post-conditions, but you may find it notationally convenient.

Proving `max` correct, relative to specifications

```
1 def max(A):
2   if len(A) == 1:
3     return A[0]
4   max_tail = max(A[1:])
5   if A[0] > max_tail:
6     return A[0]
7   else:
8     return max_tail
```

IS Let $n \in \mathbb{N}^+$, assume $P(n)$

Let A be a list of length $n+1$

Assume Pre(A)

Since $n+1 \geq 2$, we reach l4

1. len(A[1:]) = n
2. Pre(A[1:]) — all comparable; len > 0, n > 0

So, by P(n) and 1+2, Post(A[1:]), so $\forall i \; 1 \leq i < n+1 \Rightarrow$ A[i] ≤ max_tail

Case 1: max_tail ≥ A[0]

by code, we return max_tail

∴

- max_tail ≥ every ele in A ?
  - by IH max_tail ≥ every ele in A[1:]
  - and max_tail ≥ A[0], by case
- max_tail in A ?
  - by IH, max_tail is in A[1:], which is a sublist of A

So Post(A)

Proving `max` correct, relative to specifications

```
1 def max(A):
2   if len(A) == 1:
3     return A[0]
4   max_tail = max(A[1:])
5   if A[0] > max_tail:
6     return A[0]
7   else:
8     return max_tail
```

Case 2: A[0] > max_tail

we return A[0]

- A[0] in A? yes
- A[0] ≥ all eles of A?
  - by IH max_tail ≥ every ele in A[1:], by case and transitivity, A[0] ≥ every ele in A[1:]
  - A[0] ≥ A[0]

So Post(A).

So Post(A) holds in all cases.

for any list A' of length n+1, Pre(A') => Post(A')

i.e. P(n+1)

Basis

let A be a single element list. By l 2-3, we return A's one element, and Post(A), so P(1)

Recipe: proving correctness of recursive programs

1. Determine the program's **preconditions** (i.e. valid inputs) and **postconditions** (the 'correct' behaviour)
2. Define predicate $P(n) \approx$ 'for all inputs of size n, precondition $\implies$ postcondition'
3. Use induction to prove $\forall n \in \mathbb{N}, P(n)$
   3.1 Basis: Corresponds to the part of the program where the recursion 'bottoms out' (usually at the start of the function)
   3.2 Inductive step: Prove that, if the program does the right thing on smaller inputs, it does the right thing on the next input size.

   - May use simple induction (if the algorithm reduces problems of size n to $n-1$), or complete induction (e.g. if problems of size n are reduced to problems of size $n/2$)

Divide-and-conquer maximum

```
def maximum(A):
  if len(A) == 1:
    return A[0]
  mid = len(A) // 2
  L_max = maximum(A[:mid])
  R_max = maximum(A[mid:])
  if L_max > R_max:
    return L_max
  else:
    return R_max
```

Pre: as before
Post:
P(n):

pre and post are the 'API'
- should not include implementation details

Divide-and-conquer maximum

```
1  def maximum(A):
2    if len(A) == 1:
3      return A[0]
4    mid = len(A) // 2
5    L_max = maximum(A[:mid])
6    R_max = maximum(A[mid:])
7    if L_max > R_max:
8      return L_max
9    else:
10     return R_max
```

Proof sketch (complete induction)
Let $n \in \mathbb{N}^+$. Assume $\forall k \in \mathbb{N}, 0 < k < n \implies P(k)$. (IH)
Let A be a list of length n, and assume Pre(A). WTS: Post(A)
Case 1: $n = 1$. Post(A) holds (same reasoning as our previous basis)
Case 2: $n > 1$.
Show that our IH applies to recursive calls on lines 5-6
Use IH and outcome of check on line 7 to show that postcondition holds in either case, whether we reach line 8 or line 10.

Detail: showing that our IH applies to recursive calls

```python
def maximum(A):
  if len(A) == 1:
    return A[0]
  mid = len(A) // 2
  L_max = maximum(A[:mid])
  R_max = maximum(A[mid:])
  if L_max > R_max:
    return L_max
  else:
    return R_max
```

For convenience, let

- L = A[:mid]
- R = A[mid:]

WTS:

- $0 < len(L) < n$ ✓
- $0 < len(R) < n$ ✓

$mid = n//2 = \lfloor n/2 \rfloor \neq 0$

$len(L) = mid$ # by Python stuff

$= \lfloor n/2 \rfloor > 0$ # $n \geq 2$ (1)

$\lfloor n/2 \rfloor < n$ # $n \geq 2$ (2)

$len(R) = n - mid$

$= n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$

$0 < n - \lfloor n/2 \rfloor < n$ by (1) and (2)

Lemma

↗ Feel free to use this in your proofs

$\forall n \in \mathbb{N}, \lfloor n/2 \rfloor + \lceil n/2 \rceil = n$

Proof:

Detail: showing postcondition holds

```python
def maximum(A):
  if len(A) == 1:
    return A[0]
  mid = len(A) // 2
  L_max = maximum(A[:mid])
  R_max = maximum(A[mid:])
  if L_max > R_max:
    return L_max
  else:
    return R_max
```

- what we return is in A
  - by IH
- what we return is ≥ every ele in A
  - WLOG, assume we return L_max
    - by IH L_max ≥ every ele in the left half, and by IH + case + transitivity, L_max ≥ every ele in the right half.

without loss of generality

A very silly way to sum a list

```python
def sum(A):
  """Pre: A is a list containing
  only natural numbers.
  Post: return the sum of the
  numbers in A."""
  if len(A) == 0:
    return 0
  first = A[0]
  if first == 0:
    return sum(A[1:])
  else:
    A[0] = A[0] - 1
    return 1 + sum(A)
```

$P(n)$: For any list A of length n,
$\text{Pre}(A) \implies Post(A)$

Prove $\forall n \in \mathbb{N}, P(n)$ by induction?

P(n): for any A of length n
Pre(A) => Post(A)

$P(n) \overset{?}{\Rightarrow} P(n+1)$

Silly sum

```python
def sum(A):
  """Pre: A is a list containing
  only natural numbers.
  Post: return the sum of the
  numbers in A."""
  if len(A) == 0:
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  first = A[0]
  if first == 0:
    return sum(A[1:])
  else:
    A[0] = A[0] - 1
    return 1 + sum(A)
```

Proving the correctness of median

```
def median(A):
  """Pre: A is a non-empty list of unique ints.
  Post: returns the median of A."""
  return select(A, len(A)//2)

def select(A, k):
  """Pre: A is a non-empty list of unique ints.
  Post: returns the element that would be at
  index k if A were sorted."""
  pivot = A[0]
  tail = A[1:]
  S = [x for x in tail if x < pivot]
  L = [x for x in tail if x > pivot]
  if len(S) == k:
    return pivot
  elif len(S) > k:
    return select(S, k)
  else:
    return select(L, k - len(S) - 1)
```

Proving the correctness of median

```python
def median(A):
  """Pre: A is a non-empty list of unique ints.
  Post: returns the median of A."""
  return select(A, len(A)//2)

def select(A, k):
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  index k if A were sorted."""
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  S = [x for x in tail if x < pivot]
  L = [x for x in tail if x > pivot]
  if len(S) == k:
    return pivot
  elif len(S) > k:
    return select(S, k)
  else:
    return select(L, k - len(S) - 1)
```