[With post-lecture annotations]

CSC236 winter 2020, week 5: The Master Theorem

Recommended supplementary reading: David Liu 236 course notes pp 27-41, Ch. 5
"Algorithm Design" by Kleinberg & Tardos, Ch. 3 Vassos course notes

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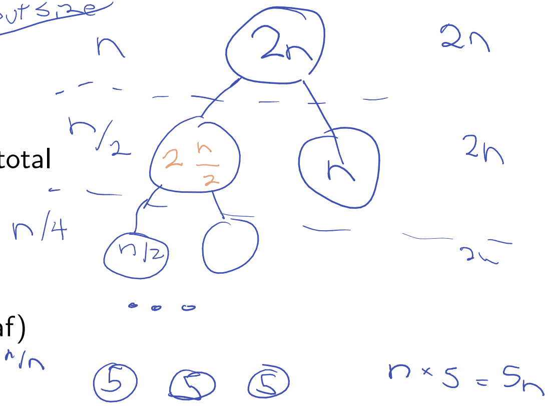
Recap: unwinding

$T(n) = 2n\log n + 5n$

$$T(n) = \begin{cases} \cancel{2}\ 5 & \text{if } n = 1 \\ 2n + 2T(n/2) & \text{if } n > 1 \end{cases}$$

Input size

Σ at level



$T(n) = 2n + 2n + 2n \ldots n \times T(\frac{n}{n})$

Convention used in slides:

- label nodes with number of non-recursive steps
- label *levels* with problem size and total steps

Also note:

- **height** $\neq$ num 'levels'
- Usually good to draw the final (leaf) level, especially if seeking an exact closed form

n/n

max distance from root to leaf

From last week: `closest_pair`

$T(n) = aT(\frac{n}{b}) + f(n)$. What are a, b, and $f(n)$?

```
def closest_distance(A):
  if len(A) == 2:
    return abs(A[0] - A[1])
  mid = len(A)//2
  L = A[:mid]
  R = A[mid:]
  # Find the closest distance between pairs that straddle L and R
  closest_LR = infinity
  for l in L:
    for r in R:
      closest_LR = min(closest_LR, abs(l-r))
  # Closest pair is either within L, within R, or between L and R
  return min(closest_LR, closest_distance(L), closest_distance(R))
```

$a = 2$ $b = 2$ $f(n) \in \Theta(n^2)$

Closed form when cost per recursive call is quadratic?

$T(n) = 2T(n/2) + f(n)$, where $f(n) \in \Theta(n^2)$

$f(n) = n^2/4 \approx n^2$

Size of problem

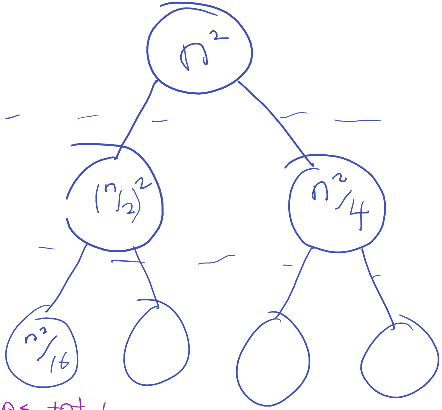
n

$n/2$

$n/4$

$n/k \Rightarrow n^2/k$ steps total

...



Total steps

n^2

$n^2/4 \times 2 = n^2/2$

$n^2/4$

$n^2/8$

...

$T(n) \cong 2n^2$

Closed form when cost per recursive call is quadratic?

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ n^2 + 2T(n/2) & \text{if } n > 1 \end{cases}$$

n leaves

$\times\ T(n/n) = 1$ steps

$$T(n) = n^2 + n^2/2 + n^2/4 + \dots n$$

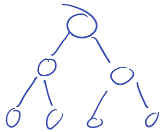
$$= \sum_{i=0}^{\log n} \frac{n^2}{2^i}$$

$$= n^2 \left(\sum_{i=0}^{\log n} \frac{1}{2^i} \right)$$

$$= n^2 \left(\overset{2}{\cancel{1}} - \frac{1}{2^{\log n}} \right) = n^2 \left(\overset{2}{\cancel{1}} - \frac{1}{n} \right) = n^2 - n$$

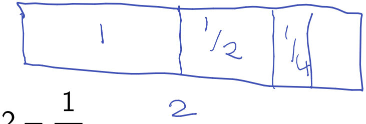
Useful geometric series to recognize

Powers of 2 come up a lot in computer science!



$$\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$$

(Number of nodes in a binary tree of height n)



$$\sum_{i=0}^{n} 2^{-i} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

But you don't *need* to memorize these. For tests, we'll either provide you with the formula, or allow you to leave these as un-reduced Σ sums.

Finding the maximum by divide-and-conquer

$f(n) \in \Theta(1)$

$T(n) = aT(\frac{n}{b}) + f(n)$. What are a, b, and $f(n)$?

$a = 2$ $b = 2$ $f(n) = 1$

```
def maximum(A):
  if len(A) == 1:
    return A[0]
  mid = len(A) // 2
  L_max = maximum(A[:mid])
  R_max = maximum(A[mid:])
  if L_max > R_max:
    return L_max
  else:
    return R_max
```

(Brainteaser: can you prove that *any* algorithm solving this problem must be in $O(n)$?)

Asymptotic runtime of `maximum`?

$T(n) = 2T(n/2) + f(n)$, where $f(n) \in \Theta(1)$

$T(n) \in \Theta(n)$

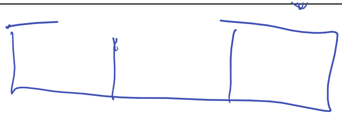
$$T(n) = 1 + 2T(n/2)$$
$$= 1 + 2(1 + 2T(n/4))$$
$$= 1 + 2(1 + 2(1 + 2T(n/8)))$$
$$= 1 + 2 + 4 + 8T(n/8)$$
$$= 1 + 2 + 4 + \ldots \quad n \cdot T(n/n)$$
$$= 1 + 2 + 4 + \ldots\ldots n$$

$$= \sum_{i=0}^{\log n} 2^i$$
$$= 2^{\log n + 1} - 1$$
$$= 2 \cdot 2^{\log n} - 1$$
$$= 2n - 1$$

Bifurcate? No, trifurcate!

$T(n) = aT(\frac{n}{b}) + f(n)$. What are a, b, and $f(n)$?

$a = 3$ $b = 3$ $f(n) = 1$

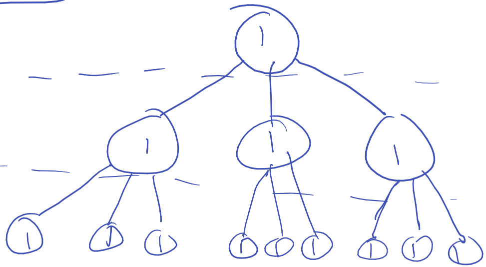


```python
def max_tri(A):
  if len(A) == 1:
    return A[0]
  m1 = len(A) // 3
  m2 = (2*len(A)) // 3
  L_max = max_tri(A[:m11])
  Centre_max = max_tri(A[m_1:m_2])
  R_max = max_tri(A[m_2:])
  if L_max > Centre_max and L_max > R_max:
    return L_max
  elif Centre_max > R_max:
    return Centre_max
  else:
    return R_max
```

Asymptotic runtime of max_tri?

$f(n) = 1$

$T(n) = 3T(n/3) + f(n)$, where $f(n) \in \Theta(1)$



Input size	Total steps at level
n	1
$n/3$	3
$n/9$	9
...	
$n/3^k$	3^k
...	
$1 = n/n$	n

[Derivation on next slide]

$$T(n) = 1 + 3 + 9 + \dots n$$

$$= \frac{3^{\log_3 n + 1} - 1}{2}$$

$$= (3n-1)/2$$

$$S = 1 + 3 + 9 + \dots 3^k$$

$$3S = 3 + 9 + \dots 3^{k+1}$$

$$3S = S - 1 + 3^{k+1}$$

$$2S = 3^{k+1} - 1$$

$$S = \frac{3^{k+1} - 1}{2}$$

$$T(n) = \begin{cases} 1, & n = 1 \\ 1 + 3T(n/3), & n > 1 \end{cases}$$

Hypothesis: $T(n) = \frac{3n-1}{2}$

n	$\frac{3n-1}{2}$	$T(n)$
1	$\frac{2}{2} = 1$	1
3	$\frac{9-1}{2} = 4$	$1 + 3T(1) = 4$
9	$\frac{26}{2} = 13$	$1 + 3T(3) = 1 + 3 \times 4 = 13$

What if $a > b$?

i.e. number of recursive calls is greater than shrinkage factor. Overlapping subproblems?

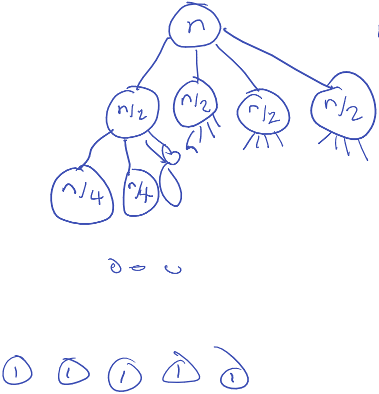
Input size

n

$n/2$

$n/4$

n/n



$$T(n) = 4T(\frac{n}{2}) + n$$

n $\quad \Sigma$

$4 \times \frac{n}{2} = 2n$

$16 \times \frac{n}{4} = 4n$

$64 \times \frac{n}{8} = 8n$

$= n \times n$

Pattern: for input size n/k, total work at that level is $k \cdot n$

$$T(n) = n + 2n + 4n + \dots n^2$$

$$= \sum_{i=0}^{\log n} 2^i n$$

$$= n \cdot \left(\sum_{i=0}^{\log n} 2^i \right)$$

$$= n \times \left(2^{\log n + 1} - 1 \right)$$

$$= n \times (2n - 1)$$

$$= 2n^2 - n$$

The Master Theorem

Now that we're thoroughly tired of unwinding...

A handy-dandy recipe for finding the asymptotic complexity of divide-and-conquer algorithms. Given $T(n)$ of the form

$$T(n) = aT(\frac{n}{b}) + f(n)$$

$a \in \mathbb{N}^+$

$b \in \mathbb{N}^+ - \{1\}$

$d \in \mathbb{R}$

The Master Theorem says that, if $f \in \Theta(n^d)$, then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_b n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Looking back

$a \stackrel{?}{=} b^d$

Algo	a	b	$f(n) \in \Theta(n^d)$	b^d	$T(n) \in \Theta(\_)$	
`mergesort`	2	2	n^1	2	$n \log n$	$=$
`closest_distance`	2	2	n^2	4	n^2	$<$
`binsearch`	1	2	$1 = n^0$	1	$n^0 \log_2 n = \log n$	$=$
`maximum`	2	2	$1 = n^0$	1	$n^{\log a} = n^{\log_2 2} = n$	$>$
`max_tri`	3	3	$1 = n^0$	1	$n^{\log_3 3} = n$	$>$
(anon)	4	2	n^1	2	$n^{\log_2 4} = n^2$	$>$

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d) \xrightarrow{\text{Master Theorem}} T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_b n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

When filling in the table on the previous slide, we observed the following pattern:
- When a > b^d, the work done at the root (i.e. the initial call) dominates the big-Theta, because the total steps at lower levels in the tree decreases exponentially
- When a < b^d, we have the opposite situation. The total big-Theta runtime is dominated by the final leaf layer. The cost at the root is very small, but it increases exponentially.
- When a = b^d, the work done at each level of the tree is exactly the same. So the big-Theta runtime is equal to the work done per level (which is n^d) multiplied by the number of levels (which is about log_b(n)).

a (the number of recursive calls) tells us the rate at which the number of nodes increases from layer to layer. b^d tells us the rate at which the work done per node shrinks as we go from layer to layer. When these are in balance, the amount of work done per level is static. Otherwise, it either grows or decreases exponentially.

Looking back even further

$$T(n) = aT(\frac{n}{b}) + \Theta(n^d) \xrightarrow{\text{Master Theorem}} T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_b n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

What about `fact`, which had recurrence

$$T(n) = 1 + T(n-1)$$

Or `subset_sum`?

$$T(n) = 1 + 2T(n-1)$$

Master Theorem can't replace unwinding for *all* recurrences. (It also doesn't give an exact closed form.)

Appendix: Slices and step counting

[Read only if you're curious]

What is the cost of running the following code?

```
# Sublist with the left half of A
L = A[:len(A)//2]
```

Reality of Python's implementation $= \Omega(n)$
In this course, we'll count it as $\Theta(1)$. Justification:

- We can generally rewrite our algorithms to avoid slicing by passing additional arguments, representing start and end indices into the original list (see next slide)
- We could also imagine our algorithms are taking `numpy` arrays instead of lists
- We don't want to tie ourselves to the implementation details of any particular language.

Except where we explicitly state otherwise, we will treat *all* built-in functions and operators as constant time.

Appendix: `maximum` without slices

Original

```python
def maximum(A):
  if len(A) == 1:
    return A[0]
  mid = len(A) // 2
  L_max = maximum(A[:mid])
  R_max = maximum(A[mid:])
  if L_max > R_max:
    return L_max
  else:
    return R_max
```

Transformed

```python
def maximum(A, start, end):
  if end - start == 1:
    return A[start]
  mid = (start + end) // 2
  L_max = maximum(A, start, mid)
  R_max = maximum(A, mid, end)
  if L_max > R_max:
    return L_max
  else:
    return R_max
```

Exercise: Use the Master Theorem to devise a recurrence T(n) having big-Theta complexity..

$T(n) \in \Theta(n^2 \log_5 n)$

$a = 25$ $\quad f(n) \in \Theta(n^2)$

$b = 5$

$d = 2$

$T(n) = 25\,T(n/5) + n^2$

$b^d = 5^2 = 25$

$T(n) \in \Theta(n^d \log_b n)$

$= \Theta(n^2 \log_5 n)$

Exercise: What are the possible big-Theta complexities of a divide-and-conquer recurrence where f(n) is constant? i.e. T(n) of the form

$T(n) = aT(n/b) + 1$

if $a = 1$, $T(n) \in \Theta(\log_b n) = \Theta(\log n) = \Theta(\log_3 n)$

$a > 1$

$T(n) \in \Theta(n^{\log_b a})$

constant

$\Theta(n^c)$ for any $c \in \mathbb{N}$

what if $a = 2^n$ ~~X~~ a must be fixed

$T(n) \in \Theta(1)$? No! Would need $\log_b a = 0$

1. Consider the following sketch of a divide-and-conquer algorithm $r(s)$ for reversing a string:

   (a) s is a string.

   (b) If $\text{len}(s) < 2$, return s

   (c) Else, partition s into three roughly equal parts: prefix s_1, suffix s_3, and mid-section s_2, and return $r(s_3) + r(s_2) + r(s_1)$.

   (d) You may assume that the time complexity of string concatenation of $s_3 + s_2 + s_1$ is proportional to $\text{len}(s_3) + \text{len}(s_2) + \text{len}(s_1)$

   Use the Master Theorem to find the asymptotic time complexity of function r in terms of $\text{len}(s)$. Be sure to show all the components of your analysis, including the values of a, b, and d. How does this compare to the complexity of simply copying the string elements in reverse order, using a loop?

We didn't get to this exercise or the next one. Feel free to try them on your own if you'd like to get some extra practice with the Master Theorem.

2. Describe a ternary version of MergeSort where the list segment to be sorted is divided into three (roughly) equal sub-lists, rather than two. Use the Master Theorem to find the asymptotic time complexity of your ternary MergeSort in terms of the length of the list segment being sorted, and compare/contrast it with the version we analyzed in class. Be sure to show all the components of your analysis, including the values of a, b, and d.