CSC236 winter 2020, week 2: complete induction

~~See section 1.3 of course notes~~ optional

Colin Morris
colin@cs.toronto.edu
http://www.~~teach~~.cs.toronto.edu/~colin/236/W20/

January 13, 2020

Complete induction

Another flavour needed

Every natural number greater than 1 has a prime factorization.

Try some examples

How does the factorization of 8 help with the factorization of 9?

$4 = 2 \times 2$

$5 = 5$

$6 = 2 \times 3$

$7 = 7$

$8 = 2 \times 2 \times 2$

$P(n) \Rightarrow P(n+1)$

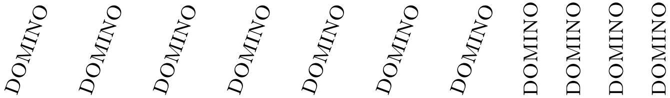
More dominoes

$P(0) \Rightarrow P(1)$

$\Rightarrow P(0)$

$P(0) \land P(1) \Rightarrow P(2)$

$P(0) \land P(1) \land P(2) \Rightarrow P(3)$



$$\overbrace{(\forall n \in \mathbb{N}, \underbrace{[\forall k \in \mathbb{N}, k < n \implies P(k)]}_{\text{Induction hypothesis}} \Rightarrow P(n))}^{\text{what we need to prove}} \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case then all cases are true

simple ind.

$P(0) \land \forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$

What about the base case?

Suppose we've proven:

$$\forall n \in \mathbb{N}, [\forall k \in \mathbb{N}, k < n \implies P(k)] \implies P(n)$$

Let $n' = 0$

Claim: $\forall k \in \mathbb{N}, k < n' \Rightarrow P(k)$

then $P(0)$

Outline of a complete induction proof

1. Define a predicate $P(n)$
2. Induction step
   2.1 Let $n \in \mathbb{N}$
   2.2 IH: Assume $\forall k \in \mathbb{N}, k < n \implies P(k)$
   2.3 Use IH to show $P(n)$

Lots of acceptable ways to write I.H.

most formal

1. Assume $\forall k \in \mathbb{N}, k < n \implies P(k)$
2. Assume $P(k)$ holds for all $k < n$
3. Assume $P(0) \wedge P(1) \wedge \ldots P(n-1)$
4. Assume $\bigwedge_{k=0}^{k=n-1} P(k)$ Σ Π
5. Assume our predicate holds for all natural numbers less than n.

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

n	3¢'s	5¢'s
8	1	1
9	3	0
10	0	2
11	2	1
12	4	0
13	1	2
14	3	1

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

$P(n)$: $\exists\, t, f \in \mathbb{N},\ n = 3t + 5f$

IS Let $n \in \mathbb{N}$, $n \geq 8$

Assume $\forall k \in \mathbb{N}$, $8 \leq k < n \Rightarrow P(k)$

Case 1: $n < 11$

$8 = 5 + 3$, $9 = 3 \cdot 3$, $10 = 5 \cdot 2$, so $P(n)$ for all such n

Case: $n \geq 11$

$n - 3 \geq 8$, so $8 \leq n - 3 < n$, so $P(n-3)$ by I.H.

let t, f, $n - 3 = 3t + 5f$

$n = 3(t+1) + 5f$, so $P(n)$. $\blacksquare$

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

$P(n)$: $3^{n+1} \geq n^3$

IS Let $n \in \mathbb{N}$, n.

Assume $\forall k \in \mathbb{N}$ $k < n \Rightarrow P(k)$ IH Assume $P(k)$, $k < n$

$P(0) \wedge \ldots P(n-1)$

Base case $n = 0$

...

$P(0)$ # cannot use IH - it says nothing

then $P(n)$

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

$P(n)$: $\exists t, f \in \mathbb{N}$, $n = 3t + 5f$

Left as a thought experiment. Is this a valid proof of $P(n), n > 7$

IS Let $n \in \mathbb{N}$, $n \geq 8$

Assume $\forall k \in \mathbb{N}$, $\cancel{8 \leq}\ k < n \Rightarrow P(k)$ # $P(0) \wedge P(1) \wedge \ldots P(n-1)$

Case 1: $n < 11$

$8 = 5 + 3$, $9 = 3 + 3 + 3$, $10 = 5 \times 2$, so $P(n)$ for all such n

Case: $n \geq 11$

$n - 3 \geq 8$, so $8 \leq n - 3 < n$, so $P(n-3)$ by I.H.

let t, f, $n - 3 = 3t + 5f$

$n = 3(t+1) + 5f$, so $P(n)$. $\blacksquare$

Postage - separate base case

Another way to structure the same proof. Not necessarily better or worse.

$P(n) : \exists t, f \in \mathbb{N}, n = 3t + 5f$

Base case:

- $P(8)$ $(t = 1, f = 1)$
- $P(9)$ $(t = 3, f = 0)$
- $P(10)$ $(t = 0, f = 2)$

Inductive step: Let $n \in \mathbb{N}$, and assume $n > 10$
Assume $\forall k \in \mathbb{N}, 8 \leq k < n \implies P(k)$
$P(n-3)$ (by IH). Let $t, f \in \mathbb{N}$, such that $n - 3 = 3t + 5f$
Then $n = 3(t+1) + 5f$, so $P(n)$

So $\forall n \in \mathbb{N}, n > 7 \implies P(n)$

Prime factorization

Show that every natural number greater than 1 has a prime factorization.

$P(n)$: n is the product of prime numbers

IS

let $n \in \mathbb{N}$, $n \geq 2$

Assume $\forall k \in \mathbb{N}$ $2 \leq k < n \Rightarrow P(k)$

Case 1: n is prime

then $P(n)$

Case 2: n is composite

let $a, b \in \mathbb{N}$

$n = a \times b$

$P(a) \wedge P(b)$ # $2 \leq a, b < n$

$a = p_1 \times p_2 \times \ldots p_j$

$b = p'_1 \times p'_2 \ldots p'_k$

$n = p_1 \times p_2 \times \ldots p'_k$

$P(n)$

then $P(n)$ in all cases.

What happens when we subtly tweak the structure?

Conventional C.I. proof structure.

0. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k < n \Rightarrow P(k)$
Blah blah blah
thus $P(n)$
So, $\forall n \in \mathbb{N}, P(n)$

Are these still valid proofs?

1. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k \leq n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n)$
Thus, by complete induction, $\forall n \in \mathbb{N}, P(n)$.

2. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k \leq n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n+1)$
Thus, by complete induction, $\forall n \in \mathbb{N}, P(n)$.

Basis: $P(0)$

$n = 1 \Downarrow P(2)$

3. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k < n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n+1)$
Thus, by complete induction, $\forall n \in \mathbb{N}, P(n)$.

$\Rightarrow P(1)$

+ Basis: $P(0)$. Valid?

$P(0) \wedge P(1) \wedge P(2) \Rightarrow P(3)$

$P(0) \wedge P(1) \wedge P(2) \Rightarrow P(4)$

A mystery recurrence

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(n \,//\, 2)]^2 + 2f(n \,//\, 2) & n > 1 \end{cases}$$

Note: In homage to Python, we'll use the notation $a \,//\, b$ to denote integer division:

$$a \,//\, b = q \iff \exists r \in \mathbb{N}, a = qb + r \wedge r < b$$

It can also be defined in terms of the floor function as $a \,//\, b = \lfloor a/b \rfloor$.

Conjecture: $f(n)$ is a multiple of 3, for $n > 1$

```
>>> 7 // 3
3
```

n	$f(n)$
0	1
1	1
2	3
3	3
4	$3^2 + 6 = 15$

For all $n > 1$, $f(n)$ is a multiple of 3?

$f(n) = \begin{cases} 1, & n \leq 1 \\ f(n//2)^2 + 2f(n//2) \end{cases}$

Use the complete induction outline

$P(n): \exists k \in \mathbb{N}, f(n) = 3k$

Let $n \in \mathbb{N}$, $n > 1$

Assume $P(2) \wedge P(3) \wedge \ldots P(n-1)$

Case 1: $n \geq 4$

$f(n) = f(n//2)^2 + 2f(n//2)$

$= (3k)^2 + 2 \times (3k)$ # by IH, $2 \leq n//_2 < n$

$= 3(3k + 2k)$

So $P(n)$

Case 2: $n < 4$

$f(2) = f(3) = 3$, so $P(n)$

So $P(n)$ holds in all cases.

Zero pair free binary strings

Denote by $Z(n)$ the number of binary strings of length n that contain no pairs of adjacent zeros. What is $Z(n)$ for the first few natural numbers n?

0

1

10

01

11

101

110

011

110101110110

What is $Z(n)$?

Use the complete induction outline

What is $Z(n)$?

Use the complete induction outline

Does binary exist?

Prove that every natural number can be written as the sum of distinct powers of 2.

$P(n)$: There exists a set of exponents $E \subset \mathbb{N}$ such that

$$n = \sum^{e \in E} 2^e$$