CSC236 winter 2020, week 2: complete induction

See section 1.3 of course notes optional

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Complete induction

Another flavour needed

Every natural number greater than 1 has a prime factorization.
Try some examples
How does the factorization of 8 help with the factorization of 9?

$4 = 2 \times 2$

$5 = 5$

$6 = 2 \times 3$

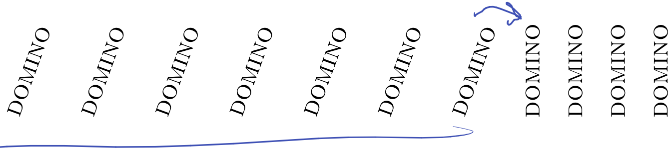
$7 = 7$

$8 = 2 \times 2 \times 2$

$P(n) \Rightarrow P(n+1)$

Simple induction
Structure doesn't work
here.

More dominoes



$$\overbrace{(\forall n \in \mathbb{N}, \underbrace{[\forall k \in \mathbb{N}, k < n \implies P(k)]}_{\text{Inductive hypothesis}} \Rightarrow P(n))}^{\text{what we have to prove}} \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case
then all cases are true

cf. simple induction $P(0) \wedge \forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$

What about the base case?

Suppose we have proved... (i.e. assume we have a complete induction proof)

$$\forall n \in \mathbb{N}, [\forall k \in \mathbb{N}, k < n \implies P(k)] \implies P(n)$$

Let n be 0

Claim: $\forall k \in \mathbb{N}, k < 0 \Rightarrow P(k)$

then $P(0)$

This is why CI doesn't need an explicit base case.

Outline of a complete induction proof

1. Define a predicate $P(n)$
2. Induction step
   2.1 Let $n \in \mathbb{N}$
   2.2 IH: Assume $\forall k \in \mathbb{N}, k < n \implies P(k)$
   2.3 Use IH to show $P(n)$

Lots of acceptable ways to write I.H.

1. Assume $\forall k \in \mathbb{N}, k < n \implies P(k)$ ← most formal
2. Assume $P(k)$ holds for all $k < n$
3. Assume $P(0) \wedge P(1) \wedge \ldots P(n-1)$
4. Assume $\bigwedge_{k=0}^{k=n-1} P(k)$ $\sum$ $\prod$
5. Assume our predicate holds for all natural numbers less than n.

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

n	3s	5s
8	1	1
9	3	0
10	0	2
11	2	1
12	4	0
13	1	2
14	3	1

$$\sum_{k=1}^{n} k$$

$$\Updownarrow$$

$$1 + 2 + \dots n$$

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

$P(n)$: $\exists t, f \in \mathbb{N},\ n = 3t + 5f$

IS

Let $n \in \mathbb{N},\ n \geq 8$

Assume $\forall k \in \mathbb{N}\ \ 8 \leq k < n \Rightarrow P(k)$ # $P(8) \wedge P(9) \wedge \ldots P(n-1)$

Case 1: $n < 11$

$8 = 5+3$, $9 = 3 \times 3$, $10 = 5 \times 2$. So in each case, $P(n)$ holds.

Case 2: $n \geq 11$

then $P(n-3)$, let $t, f \in \mathbb{N}$, such that...

$n - 3 = 3t + 5f$

$n = 3(t+1) + 5f$, then setting $t' = t+1$, $f' = f$, $P(n)$ holds

Example: postage

[Context: we talked about how this proof lets us derive $P(n)$ for some specific values of n]

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

$20 = 5 + 5 \times 3$

$P(n)$: $\exists t, f \in \mathbb{N}$, $n = 3t + 5f$

$17 = 5 + 4 \times 3$

IS

14 15 16

Let $n \in \mathbb{N}$, $n \geq 8$

Assume $\forall k$, $8 \leq k < n \Rightarrow P(k)$ # $P(8) \wedge P(9) \wedge \ldots P(n-1)$

11 12 13

Case 1: $n \leq 11$

$8 = 5 + 3$, $9 = 3 \times 3$, $10 = 5 \times 2$. So in each case, $P(n)$ holds.

Case 2: $n \geq 11$

then $P(n-3)$, let $t, f \in \mathbb{N}$, such that...

$n - 3 = 3t + 5f$

$n = 3(t+1) + 5f$, then setting $t' = t+1$, $f' = f$, $P(n)$ holds

Example: postage

$3 \times 4 + 5 \times 2 = 22$

$3 \times 5 + 5 \times 2 = 25$

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

$P(n)$: $\exists t, f \in \mathbb{N},\ n = 3t + 5f$

[Context: ???]

IS

$n = 25$

Let $n \in \mathbb{N}$, $n \geq 8$

Assume $\forall k,\ 8 \leq k < n \Rightarrow P(k)$

Case 1: $n < 11$

$8 = 5 + 3$, $9 = 3 \times 3$, $10 = 5 \times 2$. So in each case, $P(n)$ holds.

Case 2: $n \geq 11$

then $P(n-3)$ $\quad P(22)$

$n - 3 = 3t + 5f$

$n = 3(t+1) + 5f$, so $P(n)$. $\boxed{\text{QED}}$

Postage - separate base case

Another way to structure the same proof. Not necessarily better or worse.

$P(n) : \exists t, f \in \mathbb{N}, n = 3t + 5f$

Base case:

- $P(8)$ $(t = 1, f = 1)$
- $P(9)$ $(t = 3, f = 0)$
- $P(10)$ $(t = 0, f = 2)$

Inductive step: Let $n \in \mathbb{N}$, and assume $n > 10$

Assume $\forall k \in \mathbb{N}, 8 \leq k < n \implies P(k)$

$P(n-3)$ (by IH). Let $t, f \in \mathbb{N}$, such that $n - 3 = 3t + 5f$

Then $n = 3(t+1) + 5f$, so $P(n)$

So $\forall n \in \mathbb{N}, n > 7 \implies P(n)$

Prime factorization

Show that every natural number greater than 1 has a prime factorization.

$P(n)$: n is the product of prime numbers

IS

Let $n \in \mathbb{N}$, $n > 1$

Assume $\forall k \in \mathbb{N}$ $1 < k < n \Rightarrow P(k)$

Case 1: n is even, and $n \neq 2$

$n = 2k$

then $P(k)$

$P(n)$

Case 2: n is odd

$n = 2k + 1$ . . . we got stuck here, and decided to try a new approach

Prime factorization

Show that every natural number greater than 1 has a prime factorization.

$P(n)$: n is the product of prime numbers

IS

Let $n \in \mathbb{N}$, $n > 1$

Assume $\forall k \in \mathbb{N}$ $1 < k < n \Rightarrow P(k)$

Case 1: n is prime, $P(n)$

Case 2: n is composite

$\exists n_1, n_2 \in \mathbb{N}$, $n = n_1 \times n_2 \wedge 1 < n_1 < n \wedge 1 < n_2 < n$

thus $P(n_1) \wedge P(n_2)$, so n is the product of their prime f'zns

so $P(n)$

What happens when we subtly tweak the structure?

Are these still valid proofs?

Conventional structure

0. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k < \quad \Rightarrow P(k)$
Blah blah blah
thus $P(n)$
So $\forall n \in \mathbb{N}, P(n)$

X

1. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, \underline{k \leq n \Rightarrow P(k)}$
Blah blah blah...
Thus, $P(n)$
Thus, by complete induction,
$\forall n \in \mathbb{N}, P(n)$.

this includes $P(n)$
not good

2. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k \leq n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n+1)$
Thus, by complete induction,
$\forall n \in \mathbb{N}, P(n)$.

Basis: $P(0)$

2. works only if we add a separate basis of $P(0)$

3. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k \leq n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n+1)$
Thus, by complete induction,
$\forall n \in \mathbb{N}, P(n)$.

$\Downarrow$

$P(1)$ follows from this proof (by vacuous $\Rightarrow$)

+ Basis: $P(0)$

3. also works w/ added basis.

A mystery recurrence

```
def f(n):
    if n <= 1:
        return 1
    else:
        return
```

```
>>> 7 // 3
3
```

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(n \,//\, 2)]^2 + 2f(n \,//\, 2) & n > 1 \end{cases}$$

Note: In homage to Python, we'll use the notation $a \,//\, b$ to denote integer division:

$$a \,//\, b = q \iff \exists r \in \mathbb{N}, a = qb + r \wedge r < b$$

It can also be defined in terms of the floor function as $a \,//\, b = \lfloor a/b \rfloor$.

Conjecture: $f(n)$ is a multiple of 3, for $n > 1$

n	$f(n)$
0	1
1	1
2	3
3	3
4	$3^2 + 2 \times 3 = 15$
5	

For all $n > 1$, $f(n)$ is a multiple of 3?

$f(n) = \begin{cases} 1, & n \leq 1 \\ f(n//2)^2 + 2f(n//2) \end{cases}$

Use the complete induction outline

$P(n): \exists k \in \mathbb{N},\ f(n) = 3k$

Let $n \in \mathbb{N}$, $n > 1$

Assume $P(2) \wedge P(3) \wedge \ldots P(n-1)$

Case 1: $n \geq 4$

$f(n) = f(n//2)^2 + 2f(n//2)$

$= (3k)^2 + 2(3k)$ #IH, by $1 < n//2 < n$

$= 3(3k^2 + 2k)$

So $P(n)$

Case: $n < 4$

$f(2) = f(3) = 3$, so $P(n)$

So $P(n)$ holds in all cases

$n \geq 4$

$n = 4 + k$

$(4+k)//2$

$= 1 + k//2$

$g(n) = n//2$

$g(4) = 2$

Zero pair free binary strings

S = '' len(s)

Denote by $Z(n)$ the number of binary strings of length n that contain no pairs of adjacent zeros. What is $Z(n)$ for the first few natural numbers n?

'' 0101110 $\boxed{11}$

n	strings	Z(n)
0	''	1
1	1 0	2
2	01 10 11	3
3	010 011 101 110 111	5

$$f(n) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ f(n-1) + f(n-2), & n > 1 \end{cases}$$

$f(n) = Z(n)$

What is $Z(n)$?

Use the complete induction outline

$P(n): f(n) = Z(n)$

What is $Z(n)$?

Use the complete induction outline

Does binary exist?

Prove that every natural number can be written as the sum of distinct powers of 2.

$P(n)$: There exists a set of exponents $E \subset \mathbb{N}$ such that

$$n = \sum^{e \in E} 2^e$$