CSC236 winter 2020, week 2: complete induction

See section 1.3 of course notes

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Complete induction

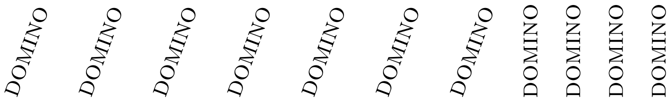
Another flavour needed

Every natural number greater than 1 has a prime factorization.

Try some examples

How does the factorization of 8 help with the factorization of 9?

More dominoes



$$(\forall n \in \mathbb{N}, [\forall k \in \mathbb{N}, k < n \implies P(k)] \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case
then all cases are true

What about the base case?

$$\forall n \in \mathbb{N}, [\forall k \in \mathbb{N}, k < n \implies P(k)] \implies P(n)$$

Outline of a complete induction proof

1. Define a predicate $P(n)$
2. Induction step
   2.1 Let $n \in \mathbb{N}$
   2.2 IH: Assume $\forall k \in \mathbb{N}, k < n \implies P(k)$
   2.3 Use IH to show $P(n)$

Lots of acceptable ways to write I.H.

1. Assume $\forall k \in \mathbb{N}, k < n \implies P(k)$
2. Assume $P(k)$ holds for all $k < n$
3. Assume $P(0) \wedge P(1) \wedge \ldots P(n-1)$
4. Assume $\bigwedge_{k=0}^{k=n-1} P(k)$
5. Assume our predicate holds for all natural numbers less than n.

Example: postage

Show that any postage amount greater than 7 cents can be formed by combining 3 and 5 cent stamps.

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Postage - separate base case

Another way to structure the same proof. Not necessarily better or worse.

$P(n) : \exists t, f \in \mathbb{N}, n = 3t + 5f$

Base case:

- $P(8)$ $(t = 1, f = 1)$
- $P(9)$ $(t = 3, f = 0)$
- $P(10)$ $(t = 0, f = 2)$

Inductive step: Let $n \in \mathbb{N}$, and assume $n > 10$
Assume $\forall k \in \mathbb{N}, 8 \leq k < n \implies P(k)$
$P(n-3)$ (by IH). Let $t, f \in \mathbb{N}$, such that $n - 3 = 3t + 5f$
Then $n = 3(t+1) + 5f$, so $P(n)$

So $\forall n \in \mathbb{N}, n > 7 \implies P(n)$

Prime factorization

Show that every natural number greater than 1 has a prime factorization.

What happens when we subtly tweak the structure?

Are these still valid proofs?

1. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k \leq n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n)$
Thus, by complete induction,
$\forall n \in \mathbb{N}, P(n)$.

2. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k \leq n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n+1)$
Thus, by complete induction,
$\forall n \in \mathbb{N}, P(n)$.

3. Let $n \in \mathbb{N}$
Assume $\forall k \in \mathbb{N}, k < n \Rightarrow P(k)$
Blah blah blah...
Thus, $P(n+1)$
Thus, by complete induction,
$\forall n \in \mathbb{N}, P(n)$.

A mystery recurrence

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(n \mathbin{//} 2)]^2 + 2f(n \mathbin{//} 2) & n > 1 \end{cases}$$

Note: In homage to Python, we'll use the notation $a \mathbin{//} b$ to denote integer division:

$$a \mathbin{//} b = q \iff \exists r \in \mathbb{N}, a = qb + r \wedge r < b$$

It can also be defined in terms of the floor function as $a \mathbin{//} b = \lfloor a/b \rfloor$.

Conjecture: $f(n)$ is a multiple of 3, for $n > 1$

For all $n > 1$, $f(n)$ is a multiple of 3?

Use the complete induction outline

Zero pair free binary strings

Denote by $Z(n)$ the number of binary strings of length n that contain no pairs of adjacent zeros. What is $Z(n)$ for the first few natural numbers n?

What is $Z(n)$?

Use the complete induction outline

What is $Z(n)$?

Use the complete induction outline

Does binary exist?

Prove that every natural number can be written as the sum of distinct powers of 2.

$P(n)$: There exists a set of exponents $E \subset \mathbb{N}$ such that

$$n = \sum^{e \in E} 2^e$$