CSC236 winter 2020

theory of computation

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Help yourselves to course info sheet

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Outline

What is this course?

P'(n): Every bipartite graph on n vertices has no more than $n^2/4$ edges if n is even, or $(n^2-1)/4$ edges if n is odd.

base case: An empty bipartite graph has 0 vertices and 0 edges, and $0 \leq 0^2/4$, which verifies $P(0)$.

inductive step: Let n be an arbitrary, fixed, natural number. Assume $P'(n)$, that every bipartite graph on n vertices has no more than $n^2/4$ edges if n is even, or $(n^2-1)/4$ edges if n is even. I will show that $P'(n+1)$ follows, that every bipartite graph on $n+1$ edges has no more than $(n+1)^2/4$ edges if $n+1$ is even, or $[(n+1)^2-1]/4$ edges if $n+1$ is odd.

Let G be an arbitrary bipartite graph on $n+1$ vertices. Remove a vertex, together with its edges, from G's larger partition to produce a new bipartite graph G'. There are two possibilities, depending on whether $n+1$ is even or odd:

case $n+1$ is odd: G's smaller partition has, at most, $n/2$ vertices, so we removed at most $n/2$ edges to produce G'. $n+1$ odd means n is even, so by assumption $P(n)$, G' has at most $n^2/4$ edges, so accounting for the edges removed G had, at most:

$$\frac{n^2}{4} + \frac{n}{2} = \frac{n^2+2n}{4} \leq \frac{(n+1)^2-1}{4}$$

So $P'(n+1)$ follows in this case.

case $n+1$ is even: G's smaller partition has, at most, $(n+1)/2$ vertices, so we removed at most $(n+1)/2$ edges to produce G'. $n+1$ even means n is odd, so by assumption $P(n)$ G' has at most $(n^2-1)/4$ edges, so accounting for the edges removed G had, at most:

$$\frac{n^2-1}{4} + \frac{n+1}{2} = \frac{n^2+2n+1}{4} \leq \frac{(n+1)^2}{4}$$

So $P'(n+1)$ follows in this case.

$P'(n+1)$ follows in both possible cases ■

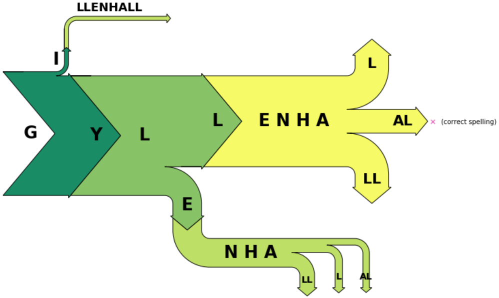
More like this...

```
    try:
        f = codecs.open(filename, "r", encoding='UTF-8')
        lines = joinLines(f.readlines())
        f.close()
    except:
        pprint("Cannot read file: %s" % filename, sys.stderr)
        sys.exit(-2)

    return lines


def scan_for_selected_frames(lines):
    """scans for frames that should be rendered exclusively,
true if such frames have been found"""
    p = re.compile("^!====\s*(.*?)\s*====(.*)", re.VERBOSE)
    for line in lines:
        mo = p.match(line)
        if mo is not None:
            return True
    return False

def line_opens_unselected_frame(line):
    p = re.compile("^====\s*(.*?)\s*====(.*)", re.VERBOSE)
    if p.match(line) is not None:
        return True
    return False

def line_opens_selected_frame(line):
```

...than this

Who am I?

Computer Science
UNIVERSITY OF TORONTO



WIKIPEDIA
The Free Encyclopedia

myfavouritethings.jpg

Who are you?

Course information sheet

- Info is a subset of what's on the course website
- Let's take a tour now
  - (Sorry, this part is boring.)

About these slides

- Adapted from Danny Heap
- Plain slides posted online in advance
- Annotated slides uploaded after lecture
  - You may want to annotate your own copy during lecture

We behave as though you already know...

- CSC165 material, especially proofs and big-Oh material
  - But you can *relax* the structure a little
- Chapter 0 material from *Introduction to Theory of Computation*
- recursion, efficiency material from CSC148

By end of course you'll know...

1. Several flavours of proof by induction
2. Reasoning about recurrences
3. Proving the correctness of programs
4. Formal languages

Simple induction

Domino fates foretold

$n \Rightarrow n+1$

DOMINO-0 DOMINO-1 DOMINO-2 DOMINO-3 DOMINO-4 DOMINO-5 DOMINO-6 DOMINO-7 DOMINO-8 DOMINO-9 DOMINO-10

$$[\,P(0)\ \wedge\ (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)\,)\,] \Longrightarrow \forall n \in \mathbb{N}, P(n)$$

IS

If the initial case works,
and each case that works implies its successor works,
then all cases work

Simple induction outline

WTS: $\forall n \in \mathbb{N},\ P(n) \Rightarrow P(n+1)$

1. Define predicate, $P(n)$
2. Inductive step
   - 2.1 Let $n \in \mathbb{N}$
   - 2.2 Assume $P(n)$ (inductive hypothesis) I.H.
   - 2.3 use it to show that $P(n+1)$ holds

   miracle occurs here
3. Verify base case(s) (AKA basis)

interchangeable

often $P(0)$

4. (optional) conclusion

Example: triangular numbers

Show that for any n, $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$.

---

1. Define predicate $P(n): \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

---

2. Inductive step Let $n \in \mathbb{N}$, assume $P(n)$

$$\sum_{k=0}^{n+1} k = \left(\sum_{k=0}^{n} k\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{\# by IH,}$$

$$= \text{''} \quad \text{''} \quad + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2},$$ thus $P(n+1)$

---

3. Base case

$$\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2},$$ so $P(0)$

4. $\forall n \in \mathbb{N}, P(n)$

Example: triangular numbers

What not to do.

Show that for any n, $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$.

---

1. Define predicate $P(n): \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

$P(n): \underline{\forall n \in \mathbb{N}}, \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

---

2. Inductive step Let $n \in \mathbb{N}$, assume $P(n)$

$$\sum_{k=0}^{n+1} k = \left(\sum_{k=0}^{n} k\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{\# by IH.}$$

$$= \text{''} \quad \text{''} \quad , \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2}, \text{ thus } P(n+1)$$

def f(x):
    x = 10
    ...
    ...

---

3. Base case

$\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}$, so $P(0)$

4. $\forall n \in \mathbb{N}, P(n)$

Sometimes we need more than one base case

Show that $\forall n \in \mathbb{N}, 3^n \geq n^3$

$P(n): 3^n \geq n^3$

I.S. Let $n \in \mathbb{N}$, assume $P(n)$, and $n \geq 3$

$3^{n+1} = 3 \cdot 3^n$

$\geq 3 \cdot n^3$

$= n^3 + n^3 + n^3$

$= n^3 + n \times n^2 + n \times n^2$

$\geq n^3 + 3n^2 + n \times n^2$ # $n \geq 3$ $(3n+1)$

$\geq n^3 + 3n^2 + n \times 9$ # $n^2 \geq 9$

$\geq n^3 + 3n^2 + 3n + 1$ # $6n \geq 1$, $n \geq 1$

$= (n+1)^3$. So $P(n+1)$

Basis: $n \geq 3$

WTS: $P(0) \wedge P(1) \wedge P(2) \wedge P(3)$

$3^3 = 27 \geq 27 = 3^3$, $P(3)$

$3^2 = 9 \geq 8 = 2^3$, $P(2)$

$3^1 = 3 \geq 1 = 1^3$, $P(1)$

$3^0 = 1 \geq 0 = 0^3$, $P(0)$

~~Sometimes we need more than one base case~~

~~Show that $\forall n \in \mathbb{N}, 3^n \geq n^3$~~

Welcome back to CSC 236

Reminders

- tutorial rooms posted to course website
  - please attempt exercises before Friday
- office hours today, 14:10 – 16:00 @ BA 2283
- extra course info sheets at the front.

Bases other than zero

Prove that $n! \geq n^2$ for $n >$???

n	$n!$	n^2	$n! \geq n^2$
0	1	0	✓
1	1	1	✓
2	2	4	✗
3	6	9	✗
4	24	16	✓
5	120	25	✓

Bases other than zero

Prove that $n! \geq n^2$ for $n \geq \cancel{???}$ 4

$P(n)$: $n! \geq n^2$

IS: let $n \in \mathbb{N}$, assume $P(n)$
and assume $n \geq 4$

$(n+1)! = n! \times (n+1)$

$\geq n^2 \times (n+1)$

$\geq (n+1)^2$ # by Lemma 1

thus $P(n+1)$

Basis

$4! = 24 \geq 16 = 4^2$, so $P(4)$

Lemma 1: $n^2 \geq n+1$, for $n \geq 4$

$n^2 = n \times n$

$\geq 2n$ # $n \geq 2$

$= n + n$

$\geq n+1$ # $n \geq 1$ $\square$

The units digit of any power of 7 is one of 1, 3, 7, or 9

Scratch work

n	Units(n)
1	1
7	7
49	9
343	3
???	1

$$\begin{array}{r} {\scriptstyle 2} \\ 343 \\ \times \quad 7 \\ \hline 1 \\ 3 \end{array}$$

The units digit of any power of 7 is one of 1, 3, 7, or 9

Use the simple induction outline Let $U = \{1, 3, 7, 9\}$

$P(n)$: $\exists k \in \mathbb{N},\ n = 7^k \wedge 7^k \bmod 10 \in U$ X not true of all $\mathbb{N}$

$\Rightarrow$

IS. Let $n \in \mathbb{N}$, assume $P(n)$

$P(n+1)$

The units digit of any power of 7 is one of 1, 3, 7, or 9

Use the simple induction outline

$P(n)$: $7^n \bmod 10 \in U$

Basis

$7^0 = 1$, which is $1 \bmod 10$, so $P(0)$

IS Let $n \in \mathbb{N}$, assume $P(n)$

$7^{n+1} = 7 \times 7^n$

Case 1: $7^n \% 10 = 1$

then $7 \times 7^n \% 10 = 7$

alt formulation

$7^n = 10k + 1$

$7^{n+1} = 7(10k+1) = 70k + 7$

$= 10 \times 7k + 7$

by modular arithmetic

Case 2: " " $= 3$

then $7 \times 7^n \% 10 = 1$

Case 3: " " $= 7$

" " $= 9$

Case 4: " " $= 9$

" " $= 3$

units digit of $7^{n+1} \in U$ in every case, thus $P(n+1)$

The units digit of any power of 7 is one of 1, 2, 3, 7, or 9

Use the simple induction outline

$U = \{1, 2, 3, 7, 9\}$

$P(n)$: $7^n \bmod 10 \in U$

Basis

$7^0 = 1$, which is $1 \bmod 10$, so $P(0)$

IS Let $n \in \mathbb{N}$, assume $P(n)$

$7^{n+1} = 7 \times 7^n$

Case 1: $7^n \% 10 = 1$

then $7 \times 7^n \% 10 = 7$

Case 2: $= 3$

then $7 \times 7^n \% 10 = 1$

Case 3: $= 7$

$= 9$

Case 4: $= 9$

$= 3$

Case 5: $7^n \% 10 = 2$

then $7^{n+1} \% 10 = 4$

:(

units digit of $7^{n+1} \in U$ in every case, thus $P(n+1)$

[Context: in this slide, we (unsuccessfully) attempted to modify our previous proof into a direct inductive proof of the claim from the next slide (where we add 2 to the set of possible digits.]

The units digit of any power of 7 is one of 1, **2**, 3, 7, or 9

Is the claim still true? What happens if you add this other case to the inductive step?

$Q(n)$: $7^n \bmod 10$ is in $\{1, 2, 3, 7, 9\}$

strengthening the I.H.

define $P(n)$ as before

prove $\forall n \in \mathbb{N},\ P(n)$

Observe $P(n) \Rightarrow Q(n)$

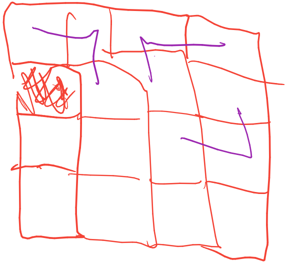
Another example:

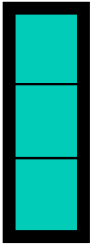
$P(n)$: $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} < 2$, $\qquad \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots < 2$

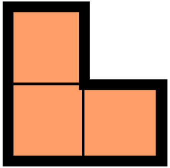
We can prove this via induction, but only if we carefully strengthen $P(n)$. Left as an exercise.

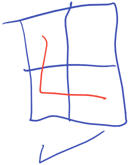
Trominoes

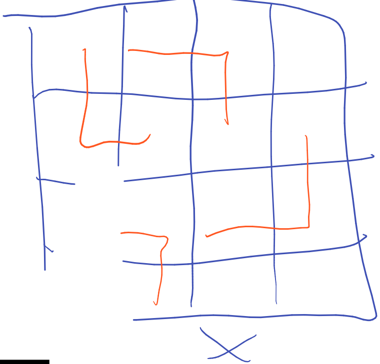
See https://en.wikipedia.org/wiki/Tromino











Can a $2^n \times 2^n$ square grid, **with one subsquare removed**, be tiled (covered without overlapping) by “chair” trominoes?

Trominoes

arbitrary?

$P(n)$: a $2^n \times 2^n$ square grid with ~~a~~ subsquare removed can be tiled with chair trominoes.

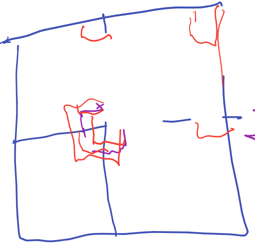
some ← top-right

Basis

P(1): 2×2 grid $\boxed{}$, P(1)

P(0): 1×1 grid □, tiled w/ 0 trominoes.

IS



2^{n+1}

Left as an exercise

Trominoes

$P(n)$: a $2^n \times 2^n$ square grid with a subsquare removed can be tiled with chair trominoes.

Note: we didn't finish this problem in lecture, though we found a good general approach for the inductive step (divide the 2^{n+1}x2^{n+1} grid into 4 subgrids, so we can apply the induction hypothesis.

We noted that the predicate as originally written was ambiguous, with a potential "weak" interpretation (the grid can be tiled with some square missing) vs. a "strong" one (the grid can be tiled with any *arbitrary* square missing). I claimed that it's possible to prove either version directly via induction. It's important that we don't exploit this ambiguity by assuming the strong version in our IH and then only proving the weak version for n+1!

Open question: once we fill in the inductive step, will it allow us to go from P(0) to P(1)? If not, that probably means we've made some implicit assumption about the n in our IH (which we should make explicit)