CSC236 winter 2020, quiz week 7_2

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Prove that the function below is partially correct with respect to the specification in the docstring. You will need to devise and prove an appropriate loop invariant, then show that, assuming the function terminates, it satisfies the postcondition.

```python
def mult(x, y):
    """Pre: x and y are ints. y is non-negative.
    Post: return x * y
    """
    p = 0
    while y > 0:
        p += x
        y -= 1
    return p
```

Lemma 1. *At the end of each iteration j*

1. $p_j + x \cdot y_j = x \cdot y_0$
2. $y_j \geq 0$

Proof. $y_0 \geq 0$ by the precondition, and $p_0 + x \cdot y_0 = 0 + x \cdot y_0$. So the invariant holds before entering the loop.

Assuming the invariant holds at the end of some iteration j, and that a $j+1$th iteration executes. Then
$p_{j+1} = p_j + x$
$y_{j+1} = y_j - 1$. Since $y_j > 0$ by the loop condition, it follows that $y_{j+1} \geq 0$.
So

$$\begin{aligned} p_{j+1} + x \cdot y_{j+1} &= (p_j + x) + (x \cdot (y_j - 1) \\ &= p_j + x \cdot y_j + x - x \\ &= p_j + x \cdot y_j \\ &= x \cdot y_0 \quad \text{\# by I.H.} \end{aligned}$$

$\square$

For arbitrary valid inputs x and y, assume the while loop of mult exits after j iterations. By the loop condition $y_j \leq 0$. By lemma 1, $y_j \geq 0$, therefore $y_j = 0$. Thus by part of 1 our invariant, $p_j + x \cdot 0 = x \cdot y_0$, thus $p_j = x \cdot y_0$, which is the correct return value.