CSC236 winter 2020, quiz week 7_1

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Prove that the function below is partially correct with respect to the specification in the docstring. i.e. that if the function terminates, it satisfies the postcondtion. You may assume without proof that at the end of the jth iteration, the mean of a_j and b_j is equal to the mean of a_0 and b_0 (hopefully this fact is intuitive from the code). You will need to find and prove an additional invariant in order to prove partial correctness.

```python
def mean(a, b):
    """Pre: a and b are integers, a < b
    Post: return the arithmetic mean of a and b (i.e. (a+b)/2)
    """
    while a < b:
        a += 1
        b -= 1
    if a != b:
        return b + .5
    else:
        return a
```

Note: for completeness, this solution includes a proof that the mean of a and b are preserved by the loop, but this was not required by the question.

Lemma 1. *At the end of each iteration j*

1. $(a_j + b_j)/2 = (a_0 + b_0)/2$
2. $a < b + 2$

Proof. Before entering the loop, part 1 holds by inspection, and part 2 follows from the precondition.

Assume the invariant holds after iteration j, and that a $j+1$th iteration executes.

$a_{j+1} = a_j + 1$ and $b_{j+1} = b_j - 1$. It follows that the average of (a_{j+1}, b_{j+1}) is the same as the average of (a_j, b_j), which, by the IH, equals the average of (a_0, b_0).

Starting from the loop condition, we know

$$\begin{aligned} a_j &< b_j \\ a_j + 1 &< b_j + 1 && \# +1 \text{ to both sides} \\ a_j + 1 &< (b_j - 1) + 2 \\ a_{j+1} &< b_{j+1} + 2 \end{aligned}$$

Satisfying part 2 of the invariant. ☐

Let a, b be arbitrary valid inputs to `mean`, and assume that the while loop exits after some iteration j. By part 1 of our invariant, the mean of a_j and b_j is equal to the mean of the original inputs, so it suffices to show that we return this.

By the loop condition $a_j \geq b_j$. By our invariant, $a_j < b_j + 2$. Since a_j and b_j are both ints[1], one of the following must be true:

Case 1: $a_j = b_j$. Then we reach line 11 and return a_j. Since $a_j = b_j$, a_j is also the mean of these two numbers, as required.

Case 2: $a_j = b_j + 1$. Then $(a_j + b_j)/2 = ((b_j + 1) + b_j)/2 = b_j + .5$, and this is exactly what we return in this case, by line 9.

[1] It wouldn't hurt to include this as an additional invariant, but for this code we consider this sufficiently "obvious" that we can get away with stating it without proof.