CSC236 tutorial exercises, Week #3
sample solutions

1. In this question, you will prove that every natural number has a unique remainder mod 10.

   (a) Use the principle of well-ordering to prove

   $$\forall n \in \mathbb{N}, \exists q, r \in \mathbb{N}, n = 10q + r \wedge r < 10$$

   Suggestion: Start by considering the set $\{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, n = 10q + r\}$.

   Solution:
   Let n be an arbitrary natural number. Consider the set

   $$R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, n = 10q + r\}$$

   R is a subset of $\mathbb{N}$ and is non-empty, since it must contain n, since $n = 10 \cdot 0 + n$. So, by the principle of well-ordering, R has a minimum element, call it r'. And let $q' \in \mathbb{N}$ be such that

   $$n = 10q' + r'$$

   For the sake of deriving a contradiction, assume that $r' \geq 10$. Thus we can write r' as $10 + k$ for some $k \in \mathbb{N}$. Thus we have

   $$\begin{aligned} n &= 10q' + r' \\ &= 10q' + (10 + k) \\ &= 10(q' + 1) + k \end{aligned}$$

   We have produced a number $k < r'$ which belongs to R, contradicting our choice of r' as the minimum element. So, by contradiction, we know $r' < 10$, as required.

   (b) *[Optional]* Now that you've shown that q and r exist, prove that they are unique. i.e. for any given $n \in \mathbb{N}$, there is only one pair of values q, r satisfying $n = 10q + r \wedge 0 \leq r < 10$.

   Solution:
   Let $n \in \mathbb{N}$, and let $q_1, r_1 \in \mathbb{N}$, such that

   $$n = 10q_1 + r_1 \wedge r_1 < 10$$

   For the sake of contradiction, let $q_2, r_2 \in \mathbb{N}$, such that $(q_2, r_2) \neq (q_1, r_1)$ and

   $$n = 10q_2 + r_2 \wedge r_2 < 10$$

Case 1: $q_1 = q_2$: Then, rearranging, we have

$$n - 10q_1 = r_1$$
$$n - 10q_1 = r_2$$

With $r_1 \neq r_2$, a contradiction.
Case 2: $q_1 \neq q_2$: Without loss of generality, assume $q_1 < q_2$. Let $k \in \mathbb{N}^+, q_2 = q_1 + k$. Then:

$$\begin{aligned} n &= 10(q_1 + k) + r_2 \\ n &= 10q_1 + 10k + r_2 \\ n - 10q_1 &= 10k + r_2 \\ r_1 &= 10k + r_2 \end{aligned}$$

Since $k > 0$, we have $r_1 \geq 10$, a contradiction.
In both cases, our assumption of multiple satisfying (q, r) pairs led to a contradiction. Therefore, (q, r) are uniquely defined for any given n.

2. Define the set of expressions $\mathcal{E}$ as the smallest set such that:

   (a) $x, y, z \in \mathcal{E}$.

   (b) If $e_1, e_2 \in \mathcal{E}$, then so are $(e_1 + e_2)$ and $(e_1 \times e_2)$.

   Define $s_1(e)$: Number of symbols from $\{(,), +, \times\}$ in e, counting duplicates.

   Define $s_2(e)$: Number of symbols from $\{x, y, z\}$ in e, counting duplicates.

   Use structural induction to prove that for all $e \in \mathcal{E}$, $s_1(e) = 3(s_2(e) - 1)$.

   Solution:
   Proof by structural induction. Define $P(e) : s_1(e) = 3(s_2(e) - 1)$.

   Base Case:Let $e \in \{x, y, z\}$. Then e has zero symbols from the set $\{(,), +, \times\}$ and one symbol (itself) from $\{x, y, z\}$. So $s_1(e) = 0 = 3(0) = 3(1 - 1) = 3(s_2(e) - 1)$, so $P(e)$ holds.

   Inductive Step:Let $e_1, e_2 \in \mathcal{E}$. Assume $P(e_1)$ and $P(e_2)$. Let $@ \in \{+, \times\}$. I will show that $P((e_1 @ e_2))$ follows.

$$\begin{aligned} s_1((e_1 @ e_2)) &= 3 + s_1(e_1) + s_1(e_2) \qquad \text{\# added two parentheses and @} \\ &= 3 + 3(s_2(e_1) - 1) + 3(s_2(e_2) - 1) \qquad \text{\# by } P(e_1) \text{ and } P(e_2) \\ &= 3((s_2(e_1) + s_2(e_2) - 1) = 3(s_2((e_1 @ e_2)) - 1) \\ &\qquad \text{\# } ((e_1 @ e_2)) \text{ has same basis symbols as } e_1 \text{ and } e_2 \text{ combined} \end{aligned}$$

   So $P((e_1 @ e_2))$ follows ■

3. Define the set of non-empty full binary trees, $\mathcal{T}$, as the smallest set such that:

   (a) Any single node is an element of $\mathcal{T}$.

   (b) If $t_1, t_2 \in \mathcal{T}$, n is a node that belongs to neither t_1 nor t_2, and t_1, t_2 have no nodes in common, then n together with edges to **the root nodes** t_1 and t_2 is also an element of $\mathcal{T}$.

Use structural induction to prove that any non-empty full binary tree has exactly one more leaf than internal nodes.

Solution:
Proof by structural induction. Define $P(t)$: t has exactly one more leaf than internal nodes.

Base Case:Let $t \in \mathcal{T}$ be a single node. Then t is itself a leaf, and has no internal nodes, and 1 is exactly one more than 0. So $P(t)$ holds.

Inductive Step:Let $t_1, t_2 \in \mathcal{T}$. Assume $P(t_1)$ and $P(t_2)$, and that t_1 and t_2 have no nodes in common. Let n be an arbitrary node that belongs to neither t_1 nor t_2, and t be the tree formed by n with edges to the roots of t_1 and t_2. I will show that $P(t)$ follows, i.e. that t has exactly one more leaf than internal nodes.

Denote the number of internal nodes and leaf nodes of t_1 by i_1 and l_1, respectively. Similarly, denote the number of internal nodes and leaf nodes of t_2 by i_2 and l_2, respectively. Notice that adding edges from n to the root nodes of these two trees does not change the status of any internal or leaf nodes, it simply adds one new internal node. Denote the number of internal and leaf nodes of t by i_t and l_t respectively, and we have:

$$\begin{aligned} i_t &= 1 + i_1 + i_2 = 1 + l_1 - 1 + l_2 - 1 \qquad \text{\# by } P(t_1) \text{ and } P(t_2) \\ &= (l_1 + l_2) - 1 = l_t - 1 \qquad \text{\# } t\text{'s leaves are exactly those of } t_1 \text{ and } t_2 \text{ combined} \end{aligned}$$

So $P(t)$ follows ■