CSC236 tutorial exercises, Week #3
best before Friday afternoon

Please attempt to solve these exercises *before* tutorial on Friday. During tutorial, your TA will answer questions about the problems, and review solutions to selected problems. These exercises are not graded, so you are encouraged to work with classmates on them, and discuss them on the course discussion board.

1. In this question, you will prove that every natural number has a unique remainder mod 10.

   (a) Use the principle of well-ordering to prove

   $$\forall n \in \mathbb{N}, \exists q, r \in \mathbb{N}, n = 10q + r \wedge r < 10$$

   Suggestion: Start by considering the set $\{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, n = 10q + r\}$.

   (b) *[Optional]* Now that you've shown that q and r exist, prove that they are unique. i.e. for any given $n \in \mathbb{N}$, there is only one pair of values q, r satisfying $n = 10q + r \wedge 0 \leq r < 10$.

2. Define the set of expressions $\mathcal{E}$ as the smallest set such that:

   (a) $x, y, z \in \mathcal{E}$.

   (b) If $e_1, e_2 \in \mathcal{E}$, then so are $(e_1 + e_2)$ and $(e_1 \times e_2)$.

   Define $s_1(e)$: Number of symbols from $\{(,), +, \times\}$ in e, counting duplicates.

   Define $s_2(e)$: Number of symbols from $\{x, y, z\}$ in e, counting duplicates.

   Use structural induction to prove that for all $e \in \mathcal{E}$, $s_1(e) = 3(s_2(e) - 1)$.

3. Define the set of non-empty full binary trees, $\mathcal{T}$, as the smallest set such that:

   (a) Any single node is an element of $\mathcal{T}$.

   (b) If $t_1, t_2 \in \mathcal{T}$, n is a node that belongs to neither t_1 nor t_2, and t_1, t_2 have no nodes in common, then n together with edges to **the root nodes** t_1 and t_2 is also an element of $\mathcal{T}$.

   Use structural induction to prove that any non-empty full binary tree has exactly one more leaf than internal nodes.