CSC236 tutorial exercises, Week #2
sample solutions

1. Full binary trees are binary trees where all internal nodes have 2 children (see page 34 of csc236 notes). Prove that any full binary tree with at least 1 node has more leaves than internal nodes. Use complete induction on the total number of nodes.

Solution:

Define $P(n)$: Every full binary tree with n nodes has more leaves than children. Proof by complete induction that $\forall n \in \mathbb{N}^{+}, P(n)$

Let $n \in \mathbb{N}^{+}$. Assume $P(k)$ holds for all $0 < k < n$. I will prove that $P(n)$ follows, that is, every full binary tree with n nodes has more leaves than children

There are three cases to consider

Case 1, $n = 1$: There is exactly one full binary tree with one node, consisting of one leaf and zero internal nodes, so $P(n)$ follows in this case.

Case 2, $n > 1$ and there are no full binary trees with n nodes: $P(n)$ is vacuously true (e.g. $n = 2$ or other even numbers greater than 0).
For the purposes of this proof, there is no need to prove the absence of all full binary trees with an even number of nodes greater than 0, although the proof is not too hard.

Case 3, $n > 1$ and one or more full binary trees with n nodes exist: Let $\mathcal{T}$ be a full binary tree with n nodes. Since $n > 1$, the root is an internal node with 2 subtrees rooted at its children, let's call them $\mathcal{T}_1$ and $\mathcal{T}_2$. Since they are children of $\mathcal{T}$, these subtrees have a positive number of nodes, we can denote these by $|\mathcal{T}_1| = n_1 > 0$ and $|\mathcal{T}_2| = n_2 > 0$. Also, since neither subtree includes the root node, we know $n > n_1, n_2$. Taking both inequalities together, we may use the inductive hypotheses $P(n_1)$ and $P(n_2)$. Denote the number of internal nodes, and the number of leaves, of $\mathcal{T}_1$ by (respectively) i_1 and l_1. Similarly, denote the leaves and internal nodes of $\mathcal{T}_2$ by i_2 and l_2. By $P(n_1)$ and $P(n_2)$, and the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are full binary trees (the degree of their internal nodes isn't changed by removing the original root), we know that $l_1 \geq i_1 + 1$ and $l_2 \geq i_2 + 1$. The number of leaves of $\mathcal{T}$, which I'll denote l, is simply the sum of those in $\mathcal{T}_1$ and $\mathcal{T}_2$, whereas the number of internal nodes of $\mathcal{T}$, which I'll denote i, is simply the sum of those in $\mathcal{T}_1$ and $\mathcal{T}_2$, plus one more (the root). Taken together I have:

$$l = l_1 + l_2 \geq i_1 + 1 + i_2 + 1 > i_1 + i_2 + 1 = i$$

So $P(n)$ follows in this case.

In every possible case $P(n)$ follows from the I.H. $\blacksquare$

2. In lecture, we proved that any postage amount greater than 7 cents can be made using a combination of 3- and 5-cent stamps. We can use the same technique to prove a similar result for 3- and 4-cent stamps. But what if 4-cent stamps are in short supply?

   (a) Use complete induction to show that postage of exactly n cents can be made using unlimited 3-cent stamps, and *at most two* 4-cent stamps, for every natural number $n > k$ (you will have to discover the value of k).

   Solution:
   Define $P(n) : \exists t, f \in \mathbb{N}, n = 3t + 4f \wedge f \leq 2$.
   Let $k = 5$. I will prove by complete induction that $\forall n \in \mathbb{N}, n > k \implies P(n)$.
   Let $n \in \mathbb{N}$ and assume $n \geq 6$. Assume $P(k)$ holds for all $k \in \mathbb{N}, 6 \leq k < n$. I will show that $P(n)$ follows, that postage of n cents can be made using unlimited 3-cent stamps and no more than two 4- cent stamps.

   Case 1, $n = 6$: $P(n)$ is satisfied by setting $t = 2, f = 0$.

   Case 2, $n = 7$: $P(n)$ is satisfied by setting $t = 1, f = 2$.

   Case 3, $n = 8$: $P(n)$ is satisfied by setting $t = 0, f = 2$.

   Case 4, $n \geq 9$: Since $9 \leq n$, $6 \leq n - 3 < n$, so we know $P(n-3)$ holds. Let $t, f \in \mathbb{N}$ be the corresponding number of 3- and 4-cent stamps for $n - 3$. Then

   $$\begin{aligned} n &= (n-3) + 3 \\ &= 3t + 4f + 3 \qquad \text{\# by I.H.} \\ &= 3(t+1) + 4f \end{aligned}$$

   Thus we can form n cents by setting $t' = t + 1, f' = f$, so $P(n)$ holds.

   So $P(n)$ follows from the I.H. in all possible cases ■

   <u>**Remark:**</u> In this case, it turns out the approach used in lecture actually requires no modifications (other than to our predicate) to satisfy the restriction on 4-cent stamps. It's worth considering how this proof would need to be adjusted if instead it was our supply of 3-cent stamps that was limited.

   (b) Prove that a single 4-cent stamp is not enough. i.e. there *does not* exist any k such that all postage amounts greater than k can be formed with unlimited 3-cent stamps and at most one 4-cent stamp. (It may be helpful to start by translating this claim into a sentence in first-order logic.)

   Solution:
   After trying several examples, it seems like it's impossible to make change for 8, 11, 14, 17, 20 etc. It seems like amounts n having $n \equiv 2 \mod 3$ are intractable. I will use this intuition to guide my proof.
   The claim we need to prove can be formally stated as

   $$\neg \left[\exists k \in \mathbb{N}, \forall n \in \mathbb{N}, n > k \implies (\exists t \in \mathbb{N}, f \in \{0,1\}, n = 3t + 4f)\right]$$

   Distributing the negation gives us

   $$\forall k \in \mathbb{N}, \exists n \in \mathbb{N}, n > k \wedge (\forall t \in \mathbb{N}, f \in \{0,1\}, n \neq 3t + 4f)$$

To prove this, let k be an arbitrary natural number. We will select an $n > k$ such that $n \equiv 2 \mod 3$. (Exactly one of $n+1$, $n+2$, or $n+3$ is guaranteed to satisfy both properties.)
For sake of contradiction, assume there exists $t \in \mathbb{N}, f \in \{0,1\}$ such that $n = 3t + 4f$.
f cannot be 0, since n is not divisible by 3, so we must have $f = 1$. Thus we have

$$\begin{aligned} n &= 3t + 4 \\ &= 3(t+1) + 1 \end{aligned}$$

Which means that $n \equiv 1 \mod 3$, a contradiction. Assuming the existence of t, f with the required properties led to a contradiction, therefore no such t and f exist. ∎

Remark: You may have been tempted to try to prove this using induction. As the above demonstrates, this is not necessary. But it is possible. The universally quantified formula above suggests a predicate of

$$P(k) : \exists n \in \mathbb{N}, n > k \wedge (\forall t \in \mathbb{N}, f \in \{0,1\}, n \neq 3t + 4f)$$

Here's a rough outline of how we could prove $\forall k \in \mathbb{N}, P(k)$. Let k be an arbitrary natural number, and assume $P(j)$ holds for all $j < k$. By inspection, I can see that 8 cannot be formed with unlimited 3-cent stamps and ≤ 1 4-cent stamps. This satisfies $P(k)$ if k is less than 8. Now, consider the case where k is at least 8. Then $P(k-1)$ is true, by our I.H. Let n be an impossible postage amount greater than $k-1$ (such an n exists, by $P(k-1)$. Then $n+3$ is greater than k. I will show $n+3$ cents cannot be formed, verifying $P(k)$. For sake of contradiction, assume $\exists x \in \mathbb{N}, y \in \{0,1\}, n+3 = 3x + 4y$. x must be greater than 1 (since $n+3 > k+2 \geq 10$, and $4y$ is at most 4). But then we can write $n = 3(x-1) + 4y$, with $x - 1 \in \mathbb{N}$, contradicting our choice of n as an impossible postage amount. By contradiction, $n+3$ is an impossible amount. So $P(k)$ holds when $k \geq 8$. $P(k)$ follows from the I.H. in all cases. ∎

3. Define function f of the natural numbers by:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2\left(f(n-2) + f(n-1)\right) + 1 & \text{if } n > 1 \end{cases}$$

Use complete induction to prove that $f(n) \leq 3^n$ for all $n \in \mathbb{N}$.

Solution:
Define $P(n) : f(n) \leq 3^n$. I will use complete induction to prove $\forall n \in \mathbb{N}, P(n)$.

Let $n \in \mathbb{N}$. Assume that $P(k)$ holds for all $k < n$. I will show that $P(n)$ follows, that is $f(n) \leq 3^n$.

Case 1, $n = 0$: Then $f(n) = 1 \leq 3^0$, so $P(n)$ follows in this case.

Case 2, $n = 1$: Then $f(n) = 3 \leq 3^1$, so $P(n)$ follows in this case.

Case 3, $n > 1$: Then

$$\begin{aligned} f(n) &= 2\left(f(n-2) + f(n-1)\right) + 1 && \# \text{ since } n \geq 2 \\ &\leq 2\left(3^{n-2} + 3^{n-1}\right) + 1 && \# \text{ by } P(n-2), P(n-1) \text{ since } n \geq 2 \Rightarrow n-2 \geq 0 \\ &= 2 \times 3^{n-2} + 2 \times 3^{n-1} + 1 \\ &= 3 \times 3^{n-2} - 3^{n-2} + 2 \times 3^{n-1} + 1 \leq 3^{n-1} - 1 + 2 \times 3^{n-1} + 1 && \# \text{ since } 3^{n-2} \geq 1 \text{ when } n > 1 \\ &= 3^n \end{aligned}$$

So $P(n)$ holds.

$P(n)$ follows from the I.H. in all possible cases. ■

4. **Bonus:** *This optional question is intended to be especially challenging. But even if you don't manage to solve it, you should make sure you understand the solution when it's posted (after tutorial).*

   Now that we've established that any amount of postage above k cents can be made with 3 and 4 cent stamps, we might naturally wonder how many different ways we can do so for a given n. We'll denote this $A(n)$. For example, $A(9) = 1$, because 9 cents can only be made one way (three 3-cent stamps), but $A(15) = 2$, because 15 can be written as $3 + 3 + 3 + 3 + 3$, or $3 + 4 + 4 + 4$.[1]

   (a) Consider the following recursively defined function:

$$B(n) = \begin{cases} 1 & \text{if } n \in \{0, 3\} \\ 0 & \text{if } n \in \{1, 2\} \\ B(n-3) + B(n-4) & \text{if } n > 3 \end{cases}$$

   Suppose we claim that $B(n) = A(n)$. Our justification is as follows: the only ways to make n cents are by making $n - 3$ cents, then adding a 3-cent stamp, or by making $n - 4$ cents, then adding a 4-cent stamp. (And the cases $\{0, 1, 2, 3\}$ are correct by inspection.)
   Find a counterexample to show that $B(n)$ *does not* correctly compute the number of ways to make n cents. What was the flaw in our reasoning?

   Solution:
   $B(7) = B(4) + B(3) = (B(1) + B(0)) + B(3) = 0 + 1 + 1 = 2$[2], however the true value of $A(7)$ is 1, because 7 cents can only be formed one way: with one 3-cent stamp, and one 4-cent stamp.
   The problem with this definition is that it's double-counting. Making 4 cents then adding a 3-cent stamp gives the same use of stamps as making 3 cents and adding a 4-cent stamp. If we cared about order (i.e. how many different ways can we place a line of stamps on an envelope such that it adds up to n?) then this formula *would* be correct.

   (b) Use complete induction to prove that the number of ways to make n cents is actually given by $A(n) = 1 + n \,//\, 3 + (-n) \,//\, 4$. (Where $//$ is the integer division operator, as defined in lecture. i.e. $a \,//\, b \equiv \lfloor a/b \rfloor$.)[3]
   If you find it helpful, you may use the following identity without proof (though the proof is not difficult): $(a + jb) \,//\, b \equiv a \,//\, b + j$, for $a, b, j \in \mathbb{Z}, b \neq 0$.

   Solution Define $P(n)$: the number of ways to make n cents using 3 and 4 cent stamps is given by $A(n) = 1 + n \,//\, 3 + (-n) \,//\, 4$.
   Let $n \in \mathbb{N}$, and assume that $P(k)$ holds for all $k < n$.

   Case 1: $n < 4$ Then our formula gives us one of $A(0) = 1$, $A(1) = 0$, $A(2) = 0$, or $A(3) = 1$. These are all correct, by inspection, so $P(n)$ holds.

[1] Note that we don't care about order, only the number of stamps of each denomination. So $3 + 4 + 4 + 4$ is the same as $4 + 3 + 4 + 4$ for our purposes.
Also, note that the 4-cent stamp shortage is no longer in effect.

[2] this is an example of a technique called *unwinding* a recurrence. We'll see a lot more of it in a couple weeks.

[3] Beware of potentially astonishing results when applying integer division / floor to negative numbers. For example, note that $(-5) \,//\, 4 = \lfloor -5/4 \rfloor = -2 \neq -(5 \,//\, 4)$. Try it out in a Python interpreter if you're incredulous.

Case 2: $n \geq 4$ Let H denote the number of ways to make n cents using only 3-cent stamps, and J denote the number of ways to make n cents using at least one 4-cent stamp. Then clearly we have $A(n) = H + J$. (By the observation that, for any combination of stamps, it either includes at least one 4-cent stamp, or it does not.)

J is equal to $A(n-4)$, since there is a 1-to-1 mapping between postage combinations forming $n-4$ cents, and postage combinations forming n cents with at least one 4-cent stamp (we can go from one set to the other by adding or removing a 4-cent stamp, respectively). So

$$\begin{aligned}
J &= A(n-4) \\
&= 1 + (n-4) \,//\, 3 + (-(n-4)) \,//\, 4 && \text{\# by I.H.} \\
&= 1 + (n-4) \,//\, 3 + (-n+4) \,//\, 4 && \text{\# simplifying} \\
&= 1 + (n-1) \,//\, 3 - 1 + (-n) \,//\, 4 + 1 && \text{\# using the rule } (a+jb) \,//\, b \equiv a \,//\, b + j \\
&= 1 + (n-1) \,//\, 3 + (-n) \,//\, 4 && \text{\# simplifying} \qquad (1)
\end{aligned}$$

Case 2a: $n \geq 4 \wedge n \equiv 0 \mod 3$. Let $k \in \mathbb{N}$, such that $n = 3k$. Substituting into (1), we have

$$\begin{aligned}
J &= 1 + (3k-1) \,//\, 3 + (-n) \,//\, 4 \\
&= 1 + k + (-1) \,//\, 3 + (-n) \,//\, 4 \\
&= 1 + k + -1 + (-n) \,//\, 4 \\
&= k + (-n) \,//\, 4 \\
&= n \,//\, 3 + (-n) \,//\, 4 && \text{\# Since } k = n \,//\, 3
\end{aligned}$$

Furthermore, since n is divisible by 3, we have $H = 1$. Thus

$$\begin{aligned}
A(n) &= H + J \\
&= 1 + n \,//\, 3 + (-n) \,//\, 4
\end{aligned}$$

As required. Thus $P(n)$.

Case 2b: $n \geq 4 \wedge n \not\equiv 0 \mod 3$. Since n is not divisible by 3, there are no ways to form n cents with only 3-cent stamps, so $H = 0$. Thus $A(n) = J$.

Let $k, r \in \mathbb{N}$, such that $n = 3k + r \wedge 0 < r < 3$. Substituting this into (1), we get

$$\begin{aligned}
A(n) = J &= 1 + (3k + r - 1) \,//\, 3 + (-n) \,//\, 4 \\
&= 1 + k + (r-1) \,//\, 3 + (-n) \,//\, 4
\end{aligned}$$

Since r is either 1 or 2, $r - 1$ is either 0 or 1, and so $(r-1) \,//\, 3$ is always 0. Furthermore, by definition of integer division, $k = n \,//\, 3$. So

$$\begin{aligned}
A(n) &= 1 + k + 0 + (-n) \,//\, 4 \\
&= 1 + n \,//\, 3 + (-n) \,//\, 4
\end{aligned}$$

Thus, $P(n)$.

$P(n)$ holds in all cases (and subcases), so our induction step is complete. ■

Remark: I could just as easily have flipped the treatment of 3 and 4 above, i.e. defining H as the number of ways to make n cents using only *4-cent stamps*, and doing sub-cases based on remainder mod 4. However, the approach we went with ends up being a little bit nicer, in that it lets us sidestep some headache-inducing reasoning about integer division of negative numbers.