CSC236 tutorial exercises, Week #2
best before Friday afternoon

Please attempt to solve these exercises *before* tutorial on Friday. During tutorial, your TA will answer questions about the problems, and review solutions to selected problems. These exercises are not graded, so you are encouraged to work with classmates on them, and discuss them on the course discussion board.

1. Full binary trees are binary trees where all internal nodes have 2 children (see page 34 of csc236 notes). Prove that any full binary tree with at least 1 node has more leaves than internal nodes. Use complete induction on the total number of nodes.

2. In lecture, we proved that any postage amount greater than 7 cents can be made using a combination of 3- and 5-cent stamps. We can use the same technique to prove a similar result for 3- and 4-cent stamps. But what if 4-cent stamps are in short supply?

   (a) Use complete induction to show that postage of exactly n cents can be made using unlimited 3-cent stamps, and *at most two* 4-cent stamps, for every natural number $n > k$ (you will have to discover the value of k).

   (b) Prove that a single 4-cent stamp is not enough. i.e. there *does not* exist any k such that all postage amounts greater than k can be formed with unlimited 3-cent stamps and at most one 4-cent stamp. (It may be helpful to start by translating this claim into a sentence in first-order logic.)

3. Define function f of the natural numbers by:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2\left(f(n-2) + f(n-1)\right) + 1 & \text{if } n > 1 \end{cases}$$

   Use complete induction to prove that $f(n) \leq 3^n$ for all $n \in \mathbb{N}$.

4. **Bonus:** *This optional question is intended to be especially challenging. But even if you don't manage to solve it, you should make sure you understand the solution when it's posted (after tutorial).*

   Now that we've established that any amount of postage above k cents can be made with 3 and 4 cent stamps, we might naturally wonder how many different ways we can do so for a given n. We'll denote this $A(n)$. For example, $A(9) = 1$, because 9 cents can only be made one way (three 3-cent stamps), but $A(15) = 2$, because 15 can be written as $3+3+3+3+3$, or $3+4+4+4$.[1]

[1] Note that we don't care about order, only the number of stamps of each denomination. So $3+4+4+4$ is the same as $4+3+4+4$ for our purposes.
Also, note that the 4-cent stamp shortage is no longer in effect.

(a) Consider the following recursively defined function:

$$B(n) = \begin{cases} 1 & \text{if } n \in \{0, 3\} \\ 0 & \text{if } n \in \{1, 2\} \\ B(n-3) + B(n-4) & \text{if } n > 3 \end{cases}$$

Suppose we claim that $B(n) = A(n)$. Our justification is as follows: the only ways to make n cents are by making $n - 3$ cents, then adding a 3-cent stamp, or by making $n - 4$ cents, then adding a 4-cent stamp. (And the cases $\{0, 1, 2, 3\}$ are correct by inspection.)

Find a counterexample to show that $B(n)$ *does not* correctly compute the number of ways to make n cents. What was the flaw in our reasoning?

(b) Use complete induction to prove that the number of ways to make n cents is actually given by $A(n) = 1 + n \,//\, 3 + (-n) \,//\, 4$. (Where // is the integer division operator, as defined in lecture. i.e. $a \,//\, b \equiv \lfloor a/b \rfloor$.)[2]

If you find it helpful, you may use the following identity without proof (though the proof is not difficult): $(a + jb) \,//\, b \equiv a \,//\, b + j$, for $a, b, j \in \mathbb{Z}, b \neq 0$.

[2] Beware of potentially astonishing results when applying integer division / floor to negative numbers. For example, note that $(-5) \,//\, 4 = \lfloor -5/4 \rfloor = -2 \neq -(5 \,//\, 4)$. Try it out in a Python interpreter if you're incredulous.